

Existential risk and growth

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Preliminary

Abstract

Human activity can create or mitigate risks of catastrophes, such as nuclear war, climate change, pandemics, or artificial intelligence run amok. These could even imperil the survival of human civilization. What is the relationship between economic growth and such existential risks? In a model of directed technical change, with moderate parameters, existential risk follows a Kuznets-style inverted U-shape. This suggests we could be living in a unique “time of perils,” having developed technologies advanced enough to threaten our permanent destruction, but not having grown wealthy enough yet to be willing to spend sufficiently on safety. Accelerating growth during this “time of perils” initially increases risk, but improves the chances of humanity’s survival in the long run. Conversely, even short-term stagnation could substantially curtail the future of humanity.

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1 Introduction

The last two centuries of economic progress have brought enormous prosperity. Yet this economic progress has also created the possibility of catastrophes such as nuclear war, extreme climate change, or bioengineered pandemics. In the next century, powerful artificial intelligence (AI) might engender amazing advances, but some worry that it could run amok. Increasing attention is being paid to these so-called “existential” risks that could imperil human civilization. In particular, existential risks are those that threaten human extinction or could otherwise irreversibly curtail the potential of humankind (such as a war that permanently sends us back to the Stone Age); see Bostrom (2002), Posner (2004), and Farquhar et al. (2017). Some philosophers argue that because of the potential “astronomical” value associated with the long-run future of humanity, mitigating existential risk that could curtail this future should be of paramount concern; see Bostrom (2003).

However, most people generally have a positive rate of pure time preference; they do not care much about the long-run future of humanity. What happens to existential risk when resources are allocated impatiently? In particular, what is the interaction between economic growth and existential risk? Does faster economic growth accelerate the development of dangerous new technologies, thereby increasing the probability of an existential catastrophe?

I develop a model of endogenous and directed technical change, involving a tradeoff between consumption and safety. Consumption and the associated technologies carry some risk of disaster, which can be mitigated by spending on safety and developing safety technology. The outcome turns out to critically depend on the scale effect of existential risk—that is, how proportionally growing both consumption and safety affects existential risk. If existential risk decreases with scale, no special concern for safety is required for risk to fall to zero exponentially and for risk to decrease with growth. If existential risk increases with scale moderately, the level of existential risk may follow an inverted U-shape. This grounds the intuition of some prominent thinkers, like Sagan (1994) and Parfit (2011), that human civilization could be passing through a unique “time of perils.” We may have advanced enough to create technologies that threaten our permanent destruction, but not yet grown wealthy enough to be willing to spend much on safety. During this “time of perils,” accelerating growth initially increases risk, but (perhaps counterintuitively) improves the chances of humanity’s survival in the long run. Conversely, even short-term stagnation substantially hurts the chances

of humanity's survival in the long run. Finally, if the scale effect of existential risk is too large and the returns to research diminish too rapidly, it is impossible to avert an eventual existential catastrophe. Though the social planner may choose the "end of growth"—stagnation in consumption—if the marginal utility of consumption diminishes rapidly, humanity is doomed to soon destruction regardless.

This paper relates to the literature on the relationship between economic growth and environmental degradation (see Brock and Taylor (2005) for an overview). Most relevant is Stokey (1998), demonstrating that if the marginal utility of consumption diminishes rapidly, there is an inverted U-shape relationship between economic development and pollution; this relationship is often called the "environmental Kuznets curve." I find that the level of existential risk may follow a similar inverted U-shape. However, Stokey (1998) looks at environmental degradation that additively reduces utility; existential risk that imperils the survival of human civilization is a quite different concern.

To model people's concern about risks of existential catastrophes, I build on previous work on the value of life. Hall and Jones (2007) show that for a large class of conventional preferences, as consumption grows and the marginal utility of consumption declines, it becomes relatively more valuable to purchase additional days of life rather than increasing consumption on any given day of life. Jones (2016) shows how this can lead society to value safety over consumption growth, resulting in optimal consumption growth lower than what is feasible. In his richer endogenous growth model, lifesaving goods can be purchased to increase people's lifespan. I build on this model to look at existential risk. Unlike the mortality in Jones's model, human activity can increase risk, and I model risk as depending on aggregate not per-capita variables.

Critically, modeling existential risk as depending on total instead of per-capita variables allows for a scale effect. Previous work has not allowed for such a scale effect. In particular, Martin and Pindyck (2015, 2019) and Aurland-Bredesen (2019) posit a fixed set of possible catastrophes, which would each require a constant permanent tax on consumption to avert. Yet this is a knife-edge assumption: holding safety spending constant as a fraction of output only holds risk constant when the scale effect is exactly zero or when population doesn't grow. This paper generalizes from this knife-edge assumption, illustrating the divergent dynamics of the cases when existential risk decreases, increases, or increases very rapidly with scale.

The rest of this paper is organized as follows. Section 2 presents the economic environment of the model and a benchmark “rule of thumb allocation.” Section 3 presents the asymptotic (impatient) optimal growth path, highlighting how the scale effect of existential risk matters for the long run. Section 4 discusses empirical evidence on the model parameters. Section 5 illustrates the transition path of the case with a moderate scale effect, yielding the inverted U-shape path of existential risk. Section 6 analyzes what happens to existential risk when growth accelerates. Section 7 presents conclusions.

2 The Economic Environment

I look at an endogenous idea-based growth model based on the Jones (1995) version of the Romer (1990) model. Similar to Jones (2016), this model features a consumption and a safety sector with directed technical change (see also Acemoglu (2002) and Acemoglu et al. (2012)).

2.1 Setup

The economy features a consumption sector, producing consumption output C_t , and a safety sector, producing safety output H_t . A different set of technology is used for each sector: A_t represents consumption technologies, while B_t represents safety technologies.

Each person is either a worker or a scientist, and in turn workers and scientists can work in either the consumption or safety sector. Workers are denoted by L and use the available technology to produce output; scientists are denoted by S and do R&D to increase the stock of technology. Total production in each sector is then given by:¹

$$C_t = A_t^\alpha L_{ct} \quad \text{and} \quad H_t = B_t^\alpha L_{ht}, \quad (1)$$

with $\alpha > 0$.

The production functions for our two sets of technologies are given by:

$$\dot{A}_t = S_{at}^\lambda A_t^\phi \quad \text{and} \quad \dot{B}_t = S_{bt}^\lambda B_t^\phi, \quad (2)$$

¹Typically, this is microfounded with a continuum of intermediate goods in each sector. I omit that here for concision. In addition, following the lead of the standard endogenous growth models, labor is the only factor of production for simplicity.

where as in Jones (1995), I assume $\phi < 1$ and $0 < \lambda \leq 1$.

Let N_t be the total population; we then have the following straightforward resource constraints:

$$L_{ct} + L_{ht} = L_t \quad \text{and} \quad S_{ct} + S_{ht} = S_t \quad \text{and} \quad L_t + S_t = N_t. \quad (3)$$

Next, consider existential risk. Jones (2016) considers individual-level mortality that can be reduced with lifesaving goods. I instead wish to consider risks that threaten the survival of humanity as a whole.

These risks differ from Jones’s individual-level mortality in two critical ways. First, these risks are man-made; humans have created the risk of catastrophic climate change, dangerous AI running amok, or nuclear war. Risk increases with more consumption: higher consumption may mean more carbon emissions, more air travel that facilitates the spread of infectious disease, more bioengineering that could result in an extremely lethal bioengineered pandemic, and more artificial intelligence technology that could go awry. At the same time, the risk of an existential catastrophe may be mitigated by investing in safety: we can abate pollution, engineer more reliable nuclear weapon locks (“permissive action links”) to reduce the risk of accidental nuclear war, or invest in pandemic preparedness. Thus, unlike in Jones (2016), where mortality is only a function of spending in the lifesaving sector, existential risk in this model is a function of both consumption and safety spending. Growth in consumption is thus not purely positive, but creates risks. This model is inspired by the idea of “differential technological development,” as articulated by Bostrom (2002): existential risk depends on the relative rate of development of potentially dangerous technologies versus technologies that ameliorate these hazards.

The second crucial difference to Jones (2016) is that existential risks depend on *total* consumption and *total* safety spending—not on per-capita consumption and per-capita safety spending. The risk of catastrophic climate change depends on total emissions, not per-capita emissions; the risk of a pandemic depends on the total amount of opportunities for a deadly (potentially bioengineered) pathogen to jump to a human (zoonosis) and then spread from there, not per-capita opportunities for zoonosis; the risk of a nuclear winter depends on the total number of nuclear weapons, not per-capita nuclear weapons; the risk from a terrorist with WMDs depends on what the craziest person is willing and able to do, not on the average person. Similarly, existential risk mitigation depends on total spending on

carbon emissions abatement, biosecurity, AI safety, etc. This introduces a scale effect: risk depends on the total size of the economy, similar to how technological development depends on scale in endogenous idea-based growth models.

I will assume that an existential catastrophe results in permanent zero utility thereafter. This assumption should be exactly valid in the case of human extinction. It should also be a valid approximation for most existential catastrophes that, while not quite killing everybody, irreversibly curtail the potential of humankind (such as a war that permanently sends us back to the Stone Age).²

Mathematically, human civilization face a time-varying hazard rate δ_t . This represents a stochastic probability of an existential catastrophe. The probability that human civilization survives to date t (starting from date 0) is given by

$$M_t = e^{-\int_0^t \delta_s ds}, \quad (4)$$

corresponding to the laws of motion

$$\dot{M}_t = -\delta_t M_t, \quad M_0 = 1. \quad (5)$$

The hazard rate is endogenous, and as explained above increases with total consumption and decreases with total safety spending:

$$\delta_t = \bar{\delta} C_t^\epsilon H_t^{-\beta}. \quad (6)$$

For those concerned about the long-run future of humanity, the key variable is $M_\infty = \lim_{t \rightarrow \infty} M_t = e^{-\int_0^\infty \delta_s ds}$. This represents the probability that human civilization does *not* succumb to an existential catastrophe and enjoys a long future with astronomical value.³ Critically, note that M_∞ is only greater than zero iff $\int_0^\infty \delta_s ds$ is bounded.

²Some have considered other risks, such as a creeping, irreversible spread of global authoritarianism; see Caplan (2008). Such a risk would not be covered by this model. I wish to focus on catastrophes that would kill most of the people alive at the time or make people's lives so miserable so as to reduce their utility to roughly zero.

³Note that surviving to time "infinity" in this model does not literally mean human civilization survives forever. Instead, it means human civilization does not destroy itself; there are other natural/physical limits to the survival of human civilization that are not considered here. In particular, there might be natural sources of extinction risk, but Snyder-Beattie et al. (2019) find that these are negligible compared to potential anthropogenic extinction risks. Thus, I focus on anthropogenic existential risk in this paper.

Given $c_t \equiv C_t/N_t$ and $h_t \equiv H_t/N_t$, expected lifetime utility for a representative agent is

$$U = \int_0^\infty e^{-\rho t} u(c_t) M_t dt, \quad (7)$$

with standard flow utility:

$$u(c_t) = \bar{u} + \frac{c_t^{1-\gamma}}{1-\gamma}. \quad (8)$$

The parameter \bar{u} is a constant that specifies the upper bound of the utility of life relative to death (with the utility of death implicitly normalized to 0) in the case where $\gamma > 1$ and thus $c^{1-\gamma}/(1-\gamma)$ is negative. See Hall and Jones (2007) for a discussion of this constant. In particular, we will generally assume that there is a level of consumption below which utility is negative, i.e. life is not worth living. For $\gamma > 1$, this means \bar{u} is positive; for $\gamma < 1$, this means \bar{u} is negative.

Finally, I assume an exogenous positive rate of population growth:⁴

$$\dot{N}_t = \bar{n} N_t. \quad (9)$$

There are then three allocative decisions that need to be made:

1. The fraction of total scientists working on consumption: $s_t \equiv \frac{S_{ct}}{S_t}$.
2. The fraction of total workers working on consumption: $\ell_t \equiv \frac{L_{ct}}{L_t}$.
3. The fraction of the population that is a scientist: $\sigma_t \equiv \frac{S_t}{N_t}$.

2.2 Rule of Thumb Allocation

As a benchmark, it will be helpful to consider a simple “rule of thumb” allocation, as in Jones (2016). This rule of thumb allocation is analogous to Solow’s (1956) assumption of a fixed saving rate in his version of the neoclassical growth model. In particular, I will consider a rule of thumb allocation where the fraction of scientists and labor working on safety is fixed. Later, I will consider the optimal allocation, in which the fraction of resources dedicated to safety can evolve.

⁴This is required for sustained exponential growth in the Jones (1995) flavor of endogenous growth models; the growing scale drives the long-run growth rate.

Proposition 1. *Balanced growth under rule of thumb allocation*

Consider a rule of thumb allocation where $s_t = \bar{s}$, $l_t = \bar{l}$, and $\sigma_t = \bar{\sigma}$, all strictly between zero and one. There exists a balanced growth path such that

$$g_A^* = g_B^* = \frac{\lambda \bar{n}}{1 - \phi}, \quad (10)$$

$$g_c^* = g_h^* = \bar{g} \equiv \frac{\alpha \lambda \bar{n}}{1 - \phi}, \quad (11)$$

$$g_\delta^* = (\epsilon - \beta)(\bar{g} + \bar{n}), \quad (12)$$

with

$$\delta_t \rightarrow 0 \text{ if } \epsilon < \beta, \quad \delta_t \rightarrow \delta^* > 0 \text{ if } \epsilon = \beta, \quad \delta_t \rightarrow \infty \text{ if } \epsilon > \beta. \quad (13)$$

Proof. See Appendix A.1. \square

Given our symmetric production functions and fixed allocation of labor and scientists to safety and consumption, the long-run growth of both sectors looks like growth in the standard Jones (1995) version of the Romer (1990) model. In particular, given the diminishing returns to research and the non-rivalry of ideas, the long-run growth of both consumption and safety depends on population growth. Moreover, given that a fixed fraction of workers and scientists is allocated to the safety sector, both safety and consumption per capita grow at the same rate.

What is important to note about this rule of thumb allocation is what happens to existential risk. In the case that $\epsilon < \beta$, i.e. safety is more potent in reducing risk than consumption is in increasing it, the hazard rate δ falls to zero at an exponential rate. Therefore, $\int_0^\infty \delta_s ds$ is bounded, which implies that the long-run probability of human civilization's survival, M_∞ , is strictly greater than zero. In the knife-edge case of $\epsilon = \beta$, the hazard rate converges to a constant, implying $M_\infty = 0$. In the case that $\epsilon > \beta$, i.e. consumption is more potent in increasing risk than safety is in decreasing it, the hazard rate increases exponentially. This causes not only $M_\infty = 0$, but in fact $\delta \rightarrow \infty$, so the instantaneous probability of an existential catastrophe approaches 1.

Here, we begin to see the central role of $\epsilon - \beta$. Recall that $\delta_t = \bar{\delta} C_t^\epsilon H_t^{-\beta}$. Thus, $\epsilon - \beta$ represents the scale effect of existential risk. If $\epsilon < \beta$, risk decreases with scale. Then, the future of humanity is bright: even if the allocation of resources to safety stays fixed, existential risk decreases exponentially. However, if $\epsilon > \beta$, existential risk increases with scale. Then, more

scale doesn't lead to more nonrival ideas and thereby more output—as it does in the classic Romer/Jones endogenous idea-based growth model—but more scale also increases risk. In the rule of thumb allocation, the fixed allocation of resources to safety leads the hazard rate to explode when $\epsilon > \beta$. In a sense, $\epsilon - \beta$ characterizes the fragility of the world.

3 The (Impatient) Optimal Allocation

I now turn to the optimal allocation. I consider a representative agent that maximizes its utility. The representative agent discounts future utility with positive rate ρ : the agent is impatient. Moreover, this representative agent is selfish, i.e. it does not consider the growing population. (However, since our population is growing at a constant rate, taking into account the growing population is equivalent to lowering ρ .)

The optimal allocation of resources is a time path for $c_t, h_t, s_t, \ell_t, \sigma_t, A_t, B_t, M_t, \delta_t$ that maximizes the utility of the representative agent, solving the following problem:

$$\max_{\{s_t, \ell_t, \sigma_t\}} U = \int_0^{\infty} M_t u(c_t) e^{-\rho t} dt, \quad (14)$$

subject to

$$c_t = A_t^\alpha \ell_t (1 - \sigma_t), \quad (15)$$

$$h_t = B_t^\alpha (1 - \ell_t) (1 - \sigma_t), \quad (16)$$

$$\dot{A}_t = s_t^\lambda \sigma_t^\lambda N_t^\lambda A_t^\phi, \quad (17)$$

$$\dot{B}_t = (1 - s_t)^\lambda \sigma_t^\lambda N_t^\lambda B_t^\phi, \quad (18)$$

$$\dot{M}_t = -\delta_t M_t, \quad \delta_t = \bar{\delta} N_t^{\epsilon - \beta} c_t^\epsilon h_t^{-\beta}. \quad (19)$$

To solve for the optimal allocation, I define the current value Hamiltonian:

$$\mathcal{H} = M_t u(c_t) + p_{at} s_t^\lambda \sigma_t^\lambda N_t^\lambda A_t^\phi + p_{bt} (1 - s_t)^\lambda \sigma_t^\lambda N_t^\lambda B_t^\phi - v_t \delta_t M_t, \quad (20)$$

where s_t, ℓ_t and σ_t are our control variables and $M_t, A_t,$ and B_t our state variables. The costate variables $p_{at}, p_{bt},$ and v_t capture the shadow values of an extra consumption idea, an extra safety idea, and an extra lifetime respectively.

Based on the maximum principle and the arguments of Romer (1986), the first-order conditions characterize a solution.

It will be useful to define

$$\tilde{v}_t \equiv \frac{v_t}{u'(c_t) c_t}. \quad (21)$$

This is the shadow value of life, converted to consumption units by $u'(c_t)$, as a ratio to the level of consumption.

After some manipulation (see Appendix A.2) the first order conditions yield:

$$\frac{1 - \ell_t}{\ell_t} = \frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t}, \quad (22)$$

$$\frac{1 - s_t}{s_t} = \frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t} \cdot \frac{\rho - g_{pat} - \phi g_{At}}{\rho - g_{pbt} - \phi g_{Bt}} \cdot \frac{g_{Bt}}{g_{At}}, \quad (23)$$

$$\frac{\sigma_t}{1 - \sigma_t} = \frac{\lambda(p_{at}\dot{A} + p_{bt}\dot{B})}{M_t[u'(c_t)c_t + (\beta - \epsilon)\delta_t v_t]}, \quad (24)$$

$$\rho = \frac{\dot{v}_t}{v_t} + \frac{1}{v_t}[u(c_t) - v_t \delta_t], \quad (25)$$

$$\rho = \frac{\dot{p}_{at}}{p_{at}} + \frac{1}{p_{at}}[M_t u'(c_t) \alpha \frac{c_t}{A_t} + p_{at} \phi \frac{\dot{A}_t}{A_t} - \alpha \epsilon v_t M_t \frac{\delta_t}{A_t}], \quad (26)$$

$$\rho = \frac{\dot{p}_{bt}}{p_{bt}} + \frac{1}{p_{bt}}[p_{bt} \phi \frac{\dot{B}_t}{B_t} + \alpha \beta v_t M_t \frac{\delta_t}{B_t}]. \quad (27)$$

The term \tilde{v}_t —and in particular the product $\delta_t \tilde{v}_t$ —thus determines the allocation of workers and scientists to consumption vs. safety. In Appendix A.2, I show that v_t can also be represented as

$$v_t = \frac{u(c_t)}{\rho - \delta_t + g_{vt}}, \quad (28)$$

and thus

$$\tilde{v}_t = \frac{\tilde{u}_t}{\rho - \delta_t + g_{vt}}, \quad \tilde{u}_t = \frac{u(c_t)}{u'(c_t) c_t}. \quad (29)$$

\tilde{u}_t is the opportunity cost of death $u(c_t)$, converted into consumption units by $u'(c_t)$, divided by the level of consumption c_t . \tilde{u} thus represents the relative value of life. The denominator of \tilde{v}_t essentially converts this into a discounted

present value. Therefore, \tilde{v}_t represents the discounted relative value of life and determines the demand for safety.

Note that the allocation of labor and scientists to safety is proportional to $\frac{\beta\delta_t\tilde{v}_t}{1-\epsilon\delta_t\tilde{v}_t}$. The numerator represents the marginal value of safety: the reduction in the hazard rate. The denominator represents the marginal value of consumption: the utility benefits of consumption (normalized to 1) minus the increase in the hazard rate. Note that $\delta_t\tilde{v}_t$ can't rise forever as in (as in Jones, 2016); if $\epsilon\delta_t\tilde{v}_t > 1$, the marginal value of consumption is negative.

3.1 The Optimal Allocation with $\epsilon \leq \beta$

First, consider the case in which safety goods are at least as potent in reducing existential risk as consumption goods in increasing existential risk, i.e. $\epsilon \leq \beta$. Then, existential risk weakly decreases with scale. The asymptotic growth path depends on the curvature of our preferences. The propositions here echo the results in Jones (2016).

Proposition 2. *Optimal growth with $\epsilon \leq \beta$ and $\gamma > 1 + (\beta - \epsilon) \left(\frac{1-\phi}{\alpha\lambda} + 1\right)$*
Assume that $\epsilon \leq \beta$ and that the marginal utility of consumption falls rapidly, in the sense that $\gamma > 1 + (\beta - \epsilon) \left(\frac{1-\phi}{\alpha\lambda} + 1\right)$. Then the optimal allocation features an asymptotic constant growth path such that as $t \rightarrow \infty$, the fraction of labor working in the consumption sector ℓ_t and the fraction of scientists working on consumption technology s_t both fall to zero at constant exponential

rates, while $\sigma_t \rightarrow \sigma^*$, and asymptotic growth is given by:⁵

$$g_A^* = \frac{\lambda(g_s + \bar{n})}{1 - \phi} > 0, \quad (30)$$

$$g_B^* = \frac{\lambda\bar{n}}{1 - \phi} > g_A^*, \quad (31)$$

$$g_h^* = \bar{g}, \quad \bar{g} \equiv \frac{\alpha\lambda\bar{n}}{1 - \phi}, \quad (32)$$

$$g_c = \bar{g} \cdot \left[\frac{\beta + (\beta - \epsilon)\frac{1-\phi}{\alpha\lambda}}{\gamma + \epsilon - 1} \right] < \bar{g}, \quad g_c^* > 0, \quad (33)$$

$$g_\delta^* = -(\gamma - 1)g_c^* < 0, \quad (34)$$

$$g_s^* = g_\ell^* = -\bar{g} \cdot \frac{\gamma - 1 - \beta + \epsilon}{(1 + \frac{\alpha\lambda}{1-\phi})(\gamma + \epsilon - 1)} - \bar{n} \cdot \frac{\epsilon - \beta}{(1 + \frac{\alpha\lambda}{1-\phi})(\gamma + \epsilon - 1)} < 0, \quad (35)$$

$$\sigma^* = \frac{\lambda\alpha g_B}{\rho + (\gamma - 1)g_c + (1 - \phi + \lambda\alpha)g_B}. \quad (36)$$

Note that $\delta_t \rightarrow 0$ exponentially, implying $M_\infty > 0$. Finally, note that this solution is valid for all $\rho > 0$.

Proof. See Appendix A.3. □

Unlike in the rule of thumb allocation, the allocation of resources to safety can adjust. In particular,

$$\tilde{u}_t = \frac{u(c_t)}{u'(c_t)c_t} = \bar{u}c_t^{\gamma-1} + \frac{1}{1 - \gamma}. \quad (37)$$

Thus, given $\gamma > 1$, the relative value of life \tilde{u}_t increases as consumption grows. As people grow wealthier, the marginal utility of consumption declines, and it becomes relatively more valuable to purchase more life and spend on avoiding death. Note that this happens regardless of discount rate ρ : no particular concern for the future is necessary for this dynamic. The rising value of life means that resources are shifted towards the safety sector. As such, consumption growth is substantially less than what is feasible and substantially less than safety growth.

⁵These results have the following form: $\lim_{t \rightarrow \infty} g_{ct} = g_c^*$, and so on.

Proposition 3. Optimal growth with $\epsilon < \beta$ and $\gamma < 1 + (\beta - \epsilon) \left(\frac{1-\phi}{\alpha\lambda} + 1\right)$
 Assume that $\epsilon < \beta$ and that the marginal utility of consumption falls, but not too rapidly, in the sense that $\gamma < 1 + (\beta - \epsilon) \left(\frac{1-\phi}{\alpha\lambda} + 1\right)$. Then the optimal allocation features an asymptotic constant growth path such that as $t \rightarrow \infty$, the fraction of labor working in the safety sector $\tilde{\ell}_t \equiv 1 - \ell_t$ and the fraction of scientists making safety ideas $\tilde{s}_t \equiv 1 - s_t$ both fall to 0 at constant exponential rates, while $\sigma_t \rightarrow \sigma^*$, and asymptotic growth is given by:

$$g_A^* = \frac{\lambda\bar{n}}{1-\phi}, \quad (38)$$

$$g_B^* = \frac{\lambda(\bar{n} + g_{\tilde{s}}^*)}{1-\phi} < g_A^*, g_B^* > 0, \quad (39)$$

$$g_c^* = \bar{g} \quad (40)$$

$$g_{\delta}^* = -\beta g_h^* + \epsilon g_c^* - (\beta - \epsilon)\bar{n} < 0, \quad (41)$$

with the exact values for $g_{\tilde{s}}^*$ and g_h^* depending on γ . If $1 < \gamma < 1 + (\beta - \epsilon) \left(\frac{1-\phi}{\alpha\lambda} + 1\right)$:

$$g_{\tilde{s}}^* = g_{\tilde{\ell}}^* = \frac{-\bar{n} \left[\frac{\alpha\lambda}{1-\phi}(1 + \beta - \epsilon - \gamma) + (\beta - \epsilon) \right]}{1 + \beta \left(1 + \frac{\alpha\lambda}{1-\phi}\right)} < 0, \quad (42)$$

$$g_h = \bar{g} \cdot \left[1 - \frac{\left(1 + \frac{\alpha\lambda}{1-\phi}\right)(1 - \gamma + \beta - \epsilon) + \left(1 + \frac{1-\phi}{\alpha\lambda}\right)(\beta - \epsilon)}{1 + \beta \left(1 + \frac{\alpha\lambda}{1-\phi}\right)} \right] < g_c^*. \quad (43)$$

If $\gamma \leq 1$:

$$g_{\tilde{s}}^* = g_{\tilde{\ell}}^* = \frac{-\bar{n} \left[\left(1 + \frac{\alpha\lambda}{1-\phi}\right)(\beta - \epsilon) \right]}{1 + \beta \left(1 + \frac{\alpha\lambda}{1-\phi}\right)} < 0, \quad (44)$$

$$g_h = \bar{g} \cdot \left[1 - \frac{\left(2 + \frac{\alpha\lambda}{1-\phi} + \frac{1-\phi}{\alpha\lambda}\right)(\beta - \epsilon)}{1 + \beta \left(1 + \frac{\alpha\lambda}{1-\phi}\right)} \right] < g_c^*. \quad (45)$$

Note that $\delta_t \rightarrow 0$ exponentially, implying $M_{\infty} > 0$.

Proof. See Appendix A.4. □

When γ is smaller, the value of life does not grow faster than the hazard rate δ_t declines. Thus, the critical product $\delta_t \tilde{v}_t$ declines, and resources are

shifted to consumption. As such, consumption growth remains as fast as is feasible.

I wish not to emphasize the difference between the allocation for larger or smaller γ , however. Instead, notice that regardless of the value of γ , the hazard rate falls exponentially to zero, and thus $M_\infty > 0$. At the same time, consumption continues to grow exponentially and $c_t \rightarrow \infty$. In that sense, the outcome of the optimal allocation in terms of the long-run future of humanity is broadly similar to the rule of thumb allocation when $\epsilon < \beta$.

To see why this is the case, note that when $\epsilon - \beta$, existential risk decreases with scale. Thus, growth naturally decreases risk.

Depending on the exact preferences, it may be possible to improve upon the rule of thumb allocation by shifting more resources to safety or to consumption over time, but the broad trajectory of the future of humanity looks bright in any case.

Finally, note that there exists a knife-edge case, which I consider for completeness.

Proposition 4. *“Interior” growth with $\epsilon < \beta$ and $\gamma = 1 + (\beta - \epsilon) \left(\frac{1-\phi}{\alpha\lambda} + 1\right)$, or with $\epsilon = \beta$ and $\gamma \leq 1$*

Assume either that $\epsilon < \beta$ and the knife-edge condition that $\gamma = 1 + (\beta - \epsilon) \left(\frac{1-\phi}{\alpha\lambda} + 1\right)$, or the knife-edge condition that $\epsilon = \beta$ and $\gamma \leq 1$. Then the optimal allocation features an asymptotic balanced growth path such that as $t \rightarrow \infty$, s_t and l_t approach constants strictly between zero and one, and the optimal allocation features the same balanced growth path as under the rule of thumb allocation.

Proof. See Appendix A.5. □

3.2 The Optimal Allocation with $\epsilon > \beta$

Now, consider the case where consumption goods are more potent in increasing existential risk than safety goods in reducing existential risk, i.e. $\epsilon > \beta$, but this difference is not too large, i.e. $\epsilon \not\gg \beta$. Again, the asymptotic growth path will depend on the curvature of our preferences. Here, we see a divergence from Jones (2016).

Proposition 5. *Optimal growth with $\epsilon > \beta$ and $\gamma > 1$*

Assume that $\epsilon > \beta$. Assume that $\epsilon \not\gg \beta$ in the sense that $\frac{\epsilon - \beta}{\beta} < \frac{\alpha\lambda}{1 - \phi}$. Finally, assume that the marginal utility of consumption falls rapidly, in the sense that

$\gamma > 1$. Then the optimal allocation features an asymptotic constant growth path such that as $t \rightarrow \infty$, the fraction of labor working in the consumption sector ℓ_t and the fraction of scientists working on consumption technology s_t both fall to zero at constant exponential rates, while $\sigma_t \rightarrow \sigma^*$, and asymptotic growth is given by:

$$g_A^* = \frac{\lambda(g_s + \bar{n})}{1 - \phi} > 0, \quad (46)$$

$$g_B^* = \frac{\lambda \bar{n}}{1 - \phi} > g_A^*, \quad (47)$$

$$g_h^* = \bar{g}, \quad (48)$$

$$g_c = \bar{g} \cdot \left[\frac{\beta + (\beta - \epsilon) \frac{1-\phi}{\alpha \lambda}}{\gamma + \epsilon - 1} \right] < \bar{g}, \quad g_c^* > 0, \quad (49)$$

$$g_\delta^* = -(\gamma - 1)g_c^* < 0, \quad (50)$$

$$g_s^* = g_\ell^* = -\bar{g} \cdot \frac{\gamma - 1 - \beta + \epsilon}{(1 + \frac{\alpha \lambda}{1-\phi})(\gamma + \epsilon - 1)} - \bar{n} \cdot \frac{\epsilon - \beta}{(1 + \frac{\alpha \lambda}{1-\phi})(\gamma + \epsilon - 1)} < 0, \quad (51)$$

$$\sigma^* = \frac{\lambda \alpha g_B}{\rho + (\gamma - 1)g_c + (1 - \phi + \lambda \alpha)g_B}. \quad (52)$$

Note that $\delta_t \rightarrow 0$ exponentially, implying $M_\infty > 0$. Finally, note that this solution is valid for all $\rho > 0$.

Proof. See Appendix A.6. □

Given $\gamma > 1$, the relative value of life \tilde{u}_t rises as consumption grows, as before. Unlike before, however, we now have $\epsilon - \beta > 0$: existential risk grows with scale. Despite this scale effect, workers and scientists are shifted to the safety sector quickly enough that δ_t still declines exponentially on the asymptotic growth path. In turn, $M_\infty > 0$. Unlike in the rule of thumb allocation, there is a nonzero probability that humanity does not succumb to an existential catastrophe.

Proposition 6. Optimal growth with $\epsilon > \beta$ and $1 - \epsilon < \gamma \leq 1$
Assume that $\epsilon > \beta$. Assume that $\epsilon \not\gg \beta$ in the sense that $\frac{\epsilon - \beta}{\beta} < \frac{\alpha \lambda}{1 - \phi}$. Assume that the marginal utility of consumption falls, but not as rapidly, in the sense that $\gamma \leq 1$. Finally, assume that the elasticity of the hazard rate is larger than the elasticity of utility with respect to consumption, i.e. $\epsilon > 1 - \gamma$. Then the optimal allocation features an asymptotic constant growth path such that

as $t \rightarrow \infty$, the fraction of labor working in the consumption sector l_t and the fraction of scientists working on consumption technology s_t both fall to zero at constant exponential rates, while $\sigma_t \rightarrow \sigma^*$, and asymptotic growth is given by:

$$g_A^* = \frac{\lambda(g_s + \bar{n})}{1 - \phi} > 0, \quad (53)$$

$$g_B^* = \frac{\lambda\bar{n}}{1 - \phi} > g_A^*, \quad (54)$$

$$g_h^* = \bar{g} \quad (55)$$

$$g_c^* = \bar{g} - (\bar{n} + \bar{g}) \frac{\epsilon - \beta}{\epsilon} < g_h^*, \quad g_c^* > 0, \quad (56)$$

$$g_\delta^* = 0, \quad (57)$$

$$\delta_t \rightarrow \frac{(1 - \gamma)\rho + (1 - \gamma)^2 g_c}{\epsilon - 1 + \gamma}, \quad (58)$$

$$g_s^* = g_\ell^* = -\frac{\epsilon - \beta}{\epsilon} \bar{n} < 0. \quad (59)$$

Note that in this case $M_\infty = 0$. Also, $\rho > (1 - \gamma)g_c$ is required for our integral to stay bounded.

Proof. See Appendix A.7. □

Unlike when $\epsilon < \beta$, workers and scientists are shifted to safety even when $\gamma \leq 1$, not just the narrower class of preferences with γ significantly greater than one as in Jones (2016). This is because even though the relative value of life \tilde{u}_t is bounded when $\gamma \leq 1$, δ_t continues increasing because of the scale effect. When $\epsilon > 1 - \gamma$, $\delta_t \tilde{v}_t$ would get too large without shifting resources, so labor and scientists are shifted to safety in the optimal allocation. Nevertheless, despite resources being shifted to safety, they are not shifted to safety quickly enough to bound $\int_0^\infty \delta_s ds$, so the long-run probability of humanity's survival is $M_\infty = 0$ when $\gamma \leq 1$.

However, there is also another case when $\epsilon < 1 - \gamma$. Here, the elasticity of the hazard rate is smaller than the elasticity of utility with respect to consumption, so resources need not be all shifted to safety to bound $\delta_t \tilde{v}_t$, and so the optimal allocation features balanced growth as in the rule of thumb allocation and $\delta_t \rightarrow \infty$.

Proposition 7. Optimal growth with $\epsilon > \beta$ and $\gamma < 1 - \epsilon$

Assume that $\epsilon > \beta$. Assume that the marginal utility of consumption falls less rapidly, in the sense that $\gamma < 1 - \epsilon$. (Note that we do NOT need to assume that $\epsilon \gg \beta$.) Then the optimal allocation features an asymptotic constant growth path such that as $t \rightarrow \infty$, $\delta_t \rightarrow \infty$, and the fraction of labor working in the consumption sector ℓ_t and the fraction of scientists making consumption ideas s_t both converge to constants strictly between 0 and 1. The optimal allocation then features a balanced growth path as under the rule of thumb allocation:

$$\ell^* = \frac{1 - \gamma - \epsilon}{1 - \gamma - \epsilon + \beta} \quad (60)$$

$$g_A^* = g_B^* = \frac{\lambda \bar{n}}{1 - \phi}, \quad (61)$$

$$g_c^* = g_h^* = \bar{g} \equiv \frac{\alpha \lambda \bar{n}}{1 - \phi}, \quad (62)$$

$$g_\delta^* = (\epsilon - \beta)(\bar{g} + \bar{n}) > 0, \quad (63)$$

$$\delta \rightarrow \infty. \quad (64)$$

Proof. See Appendix A.8. □

Critically, consider the comparison of these optimal allocations to the rule of thumb allocation. In the rule of thumb allocation when $\epsilon > \beta$, $\delta_t \rightarrow \infty$ and $M_\infty = 0$ because of the scale effect of existential risk. By contrast, when $\gamma > 1 - \epsilon$, resources are shifted to the safety sector in optimal allocation, counteracting the scale effect. Thus, δ_t converges to a small constant or even zero. Given $\gamma > 1$, the optimal allocation even features δ_t falling to zero exponentially, and thus $M_\infty > 0$; however, if $\gamma \leq 1$, $M_\infty = 0$. Nonetheless, if the marginal utility of consumption falls sufficiently slowly, i.e. $\gamma < 1 - \epsilon$, not all resources are shifted to the safety sector asymptotically and the optimal allocation looks like the rule of thumb allocation, with the hazard rate exploding.

The case of $\epsilon > \beta$ is thus a world in which existential risk is an enormous challenge, but can still be overcome. With a static concern for safety, as in the rule of thumb allocation, the scale effect portends disaster. By shifting resources to safety, as in the optimal allocation for sufficiently curved preferences, this scale effect can be contained; in fact, when $\gamma > 1$, even the impatient optimal allocation features a nonzero probability of humanity's survival in the long run.

Finally, we take care of the knife edge scenario of $\gamma = 1 - \epsilon$, which involves two subcases.

Proposition 8. Optimal growth with $\epsilon > \beta$, $\gamma = 1 - \epsilon$, and $\frac{\alpha\lambda}{1-\phi} < \frac{(\epsilon-\beta)(1+\epsilon)}{\beta\epsilon}$

Assume that $\epsilon > \beta$. Assume that the marginal utility of consumption falls, but not as rapidly, in the sense that $\gamma \leq 1$. Assume that the elasticity of the hazard rate equals than the elasticity of utility with respect to consumption, i.e. $\epsilon = 1 - \gamma$. Finally, assume that $\frac{\alpha\lambda}{1-\phi} < \frac{(\epsilon-\beta)(1+\epsilon)}{\beta\epsilon}$. Then the optimal allocation features an asymptotic constant growth path such that as $t \rightarrow \infty$, $\delta_t \rightarrow \infty$, while the fraction of labor working in the consumption sector ℓ_t and the fraction of scientists making consumption ideas s_t fall to zero exponentially. $\sigma_t \rightarrow \sigma^*$, and asymptotic growth is given by:

$$g_A^* = \frac{\lambda(g_s + \bar{n})}{1 - \phi} > 0, \quad (65)$$

$$g_B^* = \frac{\lambda\bar{n}}{1 - \phi} > g_A^*, \quad (66)$$

$$g_h^* = \bar{g}, \quad \bar{g} \equiv \frac{\alpha\lambda\bar{n}}{1 - \phi}, \quad (67)$$

$$g_c^* = \frac{1}{1 + \epsilon \left(1 + \frac{\alpha\lambda}{1-\phi}\right)} \bar{g} < \bar{g}, \quad g_c^* > 0, \quad (68)$$

$$g_\delta^* = (\epsilon - \beta)\bar{n} + \epsilon g_c - \beta g_h > 0, \quad \delta \rightarrow \infty \quad (69)$$

$$g_s^* = g_\ell^* = -\epsilon g_c < 0. \quad (70)$$

Proof. See Appendix A.9. □

Proposition 9. Optimal growth with $\epsilon > \beta$, $\gamma = 1 - \epsilon$, and $\frac{\alpha\lambda}{1-\phi} \geq \frac{(\epsilon-\beta)(1+\epsilon)}{\beta\epsilon}$

Assume that $\epsilon > \beta$. Assume that the marginal utility of consumption falls, but not as rapidly, in the sense that $\gamma \leq 1$. Assume that the elasticity of the hazard rate equals than the elasticity of utility with respect to consumption, i.e. $\epsilon = 1 - \gamma$. Finally, assume that $\frac{\alpha\lambda}{1-\phi} \geq \frac{(\epsilon-\beta)(1+\epsilon)}{\beta\epsilon}$. Then the optimal allocation features an asymptotic constant growth path such that as $t \rightarrow \infty$, the fraction of labor working in the consumption sector ℓ_t and the fraction of scientists working on consumption technology s_t both fall to zero at constant

exponential rates, while $\sigma_t \rightarrow \sigma^*$, and asymptotic growth is given by:

$$g_A^* = \frac{\lambda(g_s + \bar{n})}{1 - \phi} > 0, \quad (71)$$

$$g_B^* = \frac{\lambda\bar{n}}{1 - \phi} > g_A^*, \quad (72)$$

$$g_h^* = \bar{g}, \quad (73)$$

$$g_c^* = \bar{g} - (\bar{n} + \bar{g})\frac{\epsilon - \beta}{\epsilon} < g_h^*, \quad g_c^* > 0, \quad (74)$$

$$g_\delta^* = 0, \quad \text{but } \delta \rightarrow \infty \text{ subexponentially,} \quad (75)$$

$$g_s^* = g_\ell^* = -\frac{\epsilon - \beta}{\epsilon}\bar{n} < 0. \quad (76)$$

Proof. See Appendix A.10. □

3.3 Certain Existential Catastrophe with $\epsilon \gg \beta$

Now, consider the case in which $\epsilon \gg \beta$, i.e. consumption goods are significantly more potent in increasing existential risk than safety goods are in reducing it.

Now, there is no way to stop $\delta_t \rightarrow \infty$. To understand why this is the case, note that in the Jones (1995) version of the Romer (1990) model, growth in the long run is $\frac{\alpha\lambda}{1-\phi}\bar{n}$: the diminishing returns to R&D combined with the nonrivalry of ideas means that in the long run, the growth rate depends on the growth rate of population. In our model, even if all workers and scientists are shifted to working on safety asymptotically, the contribution of safety growth to the growth rate of δ would be $-\beta\frac{\alpha\lambda}{1-\phi}\bar{n}$.

At the same time, when $\epsilon > \beta$ in our model, existential risk increases with scale: population growth increases scale and thus contributes $(\epsilon - \beta)\bar{n}$ to the growth rate of δ . The problem arises when $(\epsilon - \beta)\bar{n} > \beta\frac{\alpha\lambda}{1-\phi}\bar{n}$: then, even if everyone were to work on safety asymptotically and even if per capita consumption were to stay constant, that cannot stop the hazard rate δ_t from growing.

When $\frac{\epsilon - \beta}{\beta} > \frac{\alpha\lambda}{1 - \phi}$, the scale effect of existential risk is larger than the scale effect of ideas. Thus, given exogenous population growth, there is no way to stop $\delta_t \rightarrow \infty$ and $M_\infty = 0$. Even halting population growth and stagnating would only provide temporary relief: without population growth, there is no growth in safety technology. δ_t would remain at a constant high

level, existential catastrophe follows eventually, and $M_\infty = 0$. Letting c_t fall exponentially could temporarily reduce existential risk, but eventually life would be so miserable that extinction would be preferable to continued existence. In short, when $\epsilon \gg \beta$, eventual existential catastrophe is inevitable regardless of what society does.

This case stands in stark contrast to the previous case when $\epsilon > \beta$. When the scale effect of existential risk is not too large, $c_t \rightarrow \infty$ and, given sufficiently curved preferences, $M_\infty > 0$. Even with a scale effect, existential risk could be overcome. When $\epsilon \gg \beta$, the scale effect of existential risk is larger than the scale effects of ideas. A key factor here is $\frac{\alpha\lambda}{1-\phi}$. If the returns to more research (ϕ) and more people working on research (λ) do not decrease as rapidly, $\frac{\alpha\lambda}{1-\phi}$ is higher, and so a larger scale effect of existential risk can be dealt with.

In some sense, the world of $\epsilon \gg \beta$ is the economist's version of the Fermi Paradox or the Doomsday Argument: the world is simply too fragile and R&D too hard for existential risk to be overcome.

Now we come to the optimal allocations. Note that we already dealt with the case where $\gamma \leq 1 - \epsilon$ in the previous section. In particular, when $\gamma < 1 - \epsilon$ (proposition 7), the asymptotic growth path features positive consumption growth while s_t and l_t converge to constants (while $\delta_t \rightarrow \infty$ with positive g_δ , as is unavoidable). Thus, when the diminishing returns to consumption fall sufficiently slowly, the relative concern for life is small enough that despite $\epsilon \gg \beta$, the optimal allocation still doesn't shift all resources to safety asymptotically and consumption grows exponentially without bound.

In fact, this is the only $\epsilon \gg \beta$ case where consumptions grows exponentially without bound. When $\gamma > 1 - \epsilon$, we have what we might call an "end of growth" scenario (see the proposition below). Risk grows so rapidly with the scale effect and the diminishing returns to consumption are large enough that the social planner chooses to eventually have consumption stagnate rather than add even more risk. Note, however, that even stagnation cannot stop $\delta_t \rightarrow \infty$ in the presence of population growth, so $M_\infty = 0$ (and as explained earlier, even stopping population growth would result in $M_\infty = 0$).

In the below, I will call c^* the level of consumption that c_t converges to and stagnates at, and I will call c_0 the level of consumption at which utility is 0. c_0 is an immiseration level of consumption; anything below that and life is not worth living.

Proposition 10. *Optimal growth with $\epsilon \gg \beta$ and $\gamma > 1 - \epsilon$*

Assume that $\epsilon \gg \beta$ in the sense that $\frac{\epsilon - \beta}{\beta} > \frac{\alpha\lambda}{1 - \phi}$. Assume that the marginal utility of consumption falls at least moderately rapidly, in the sense that $\gamma > 1 - \epsilon$. Then the optimal allocation features a constant asymptotic growth path such that as $t \rightarrow \infty$, consumption stagnates with $g_c = 0$ and $c \rightarrow c^*$, $\delta_t \rightarrow \infty$, while the fraction of labor and scientists working in the consumption sector ℓ_t and s_t fall to zero exponentially and $\sigma_t \rightarrow \sigma^*$. Asymptotic growth is given by:

$$g_A^* = \frac{\lambda(g_s + \bar{n})}{1 - \phi} > 0, \quad (77)$$

$$g_B^* = \frac{\lambda\bar{n}}{1 - \phi} > g_A^*, \quad (78)$$

$$g_h^* = \bar{g}, \quad (79)$$

$$g_c = 0, \quad c_t \rightarrow c^*, \quad (80)$$

$$g_\delta = (\epsilon - \beta)\bar{n} - \beta g_h^* > 0, \quad \delta \rightarrow \infty, \quad (81)$$

$$g_s^* = g_\ell^* = -\frac{1}{\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)} \bar{g} < 0, \quad (82)$$

with the value of c^* depending on γ . If $\gamma = 1$, i.e. utility is logarithmic in consumption,

$$c^* = \exp\left(\frac{1}{\epsilon} - \bar{u}\right), \quad (83)$$

$$\frac{c^*}{c_0} = \exp\left(\frac{1}{\epsilon}\right), \quad (84)$$

while if $\gamma > 1$ or $1 > \gamma > 1 - \epsilon$:

$$c^* = \left[\frac{\frac{1}{\epsilon} + \frac{1}{\gamma - 1}}{\bar{u}}\right]^{\frac{1}{\gamma - 1}}, \quad (85)$$

$$\frac{c^*}{c_0} = \left[\frac{\gamma - 1}{\epsilon} + 1\right]^{\frac{1}{\gamma - 1}} \quad (86)$$

Proof. See Appendix A.11. □

We can then look at the ratio of c^* relative to c_0 , i.e. how much higher the level of consumption society stagnates at is than the immiseration level of

consumption. For example, if we think an immiseration level of consumption corresponds to \$300/year, and $\frac{c^*}{c_0} = 10$, then c^* corresponds to consumption of \$3000/year.

For example, for $\epsilon = 1/2$ and $\gamma = 2$, $\frac{c^*}{c_0} = 3$, i.e. consumption converges to a level that is just three times higher than the immiseration level. For the same ϵ but $\gamma = 1$, $\frac{c^*}{c_0} \approx 7.4$, i.e. consumption converges to a level that is about 7.4 times higher than the immiseration level. And for the same ϵ but $\gamma = 3/4$, $\frac{c^*}{c_0} = 16$, i.e. consumption converges to a level that is 16 times higher than the immiseration level.

We see that $\frac{c^*}{c_0}$ decreases with γ . Similarly, $\frac{c^*}{c_0}$ decreases with ϵ . To illustrate this, we can look at a contour plot of $\frac{c^*}{c_0}$ for different levels of γ and ϵ . (For technical reasons, the plot only shows parameter combinations that result in $\frac{c^*}{c_0} \leq 100$.)

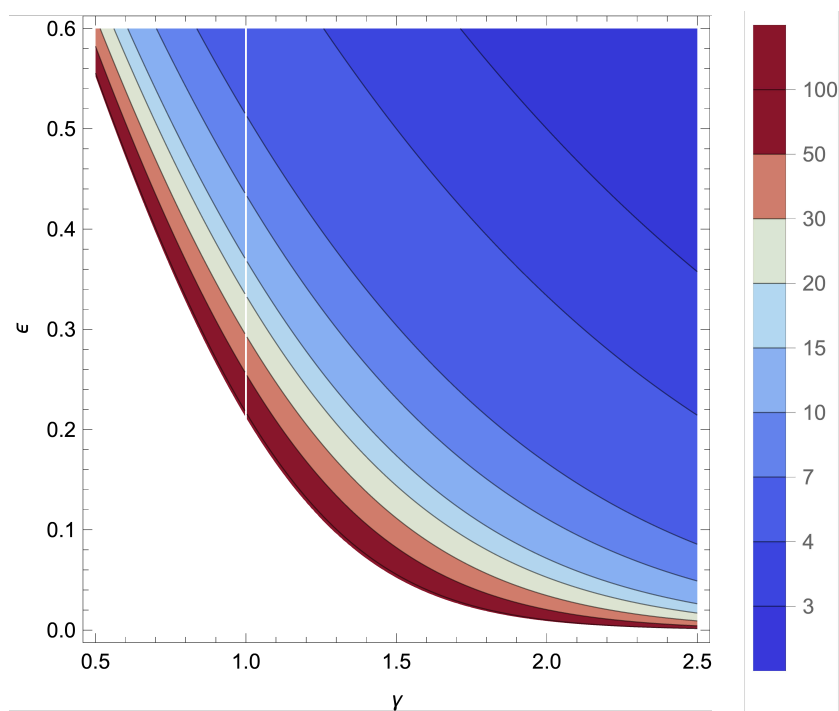


Figure 1: The ratio of the “end of growth” level of consumption, c^* , relative to the immiseration level of consumption, c_0 , for different values of γ and ϵ

This “end of growth“ scenario might accord with some common intuitions.

Perhaps economic growth and the technological innovations that come with it are just too dangerous. Instead of continuing to grow indefinitely, it might be optimal—even for an impatient society!—to halt consumption growth. (Indeed, from per-capita perspective, it could even be optimal to halt—or even reverse—population growth as well, but population growth is exogenous in our model.)

It is worth keeping in mind, though, that the world in which the “end of growth“ is optimal is also the world in which an existential catastrophe is inevitable. The “end of growth” may slow the rise of risk, but an existential catastrophe will destroy civilization soon anyway. From a “longtermist” perspective, i.e. from the perspective of total, non-discounted utilitarian welfare, is ultimately not much value in this world, since there is no chance of humanity surviving—there is no potential for “astronomical value.” Even some totalitarian state that forced population to fall exponentially and had c_t fall to just above the immiseration level in the name of risk reduction would fail to enable humanity’s survival: the last person alive couldn’t do enough R&D to have δ_t (continue) to fall exponentially, so $M_\infty = 0$.

Finally, we consider the knife-edge case for completeness.

Proposition 11. *Optimal growth with $\frac{\epsilon-\beta}{\beta} = \frac{\alpha\lambda}{1-\phi}$ and $\gamma > 1 - \epsilon$*

Assume that $\frac{\epsilon-\beta}{\beta} = \frac{\alpha\lambda}{1-\phi}$. Assume that $\gamma > 1 - \epsilon$. Then, the optimal allocation features an asymptotic growth path such that $g_c = 0$, $g_\delta = 0$, while the fraction of labor and scientists working in the consumption sector ℓ_t and s_t fall to zero exponentially. In particular, note that in the case that $\gamma \leq 1$, c_t can still increase (subexponentially) without bound while $\delta_t \rightarrow \delta^ < 1$. In the case that $\gamma \geq 1$, c_t must still be bounded.*

Proof. See Appendix A.12. □

3.4 Summary

To provide an overview of the various optimal allocations, I have compiled an overview of the asymptotic growth paths under different parameter values below. For the sake of clarity, I have omitted the knife-edge cases.

Table 1: Overview of Optimal Allocations

	$\epsilon < \beta$	$\epsilon > \beta$	$\epsilon \gg \beta$
Rule of thumb allocation	$g_c = g_h = \bar{g}$ $\delta_t \rightarrow 0$ $M_\infty > 0$	$g_c = g_h = \bar{g}$ $\delta_t \rightarrow \infty$ $M_\infty = 0$	
Optimal allocation with smallest γ	$s_t, \ell_t \rightarrow 1$ $g_c = \bar{g}, g_h < g_c$ $\delta_t \rightarrow 0$ $M_\infty > 0$	$s_t, \ell_t \rightarrow s^*, \ell^*$ $g_c = g_h = \bar{g} > 0$ $\delta_t \rightarrow \infty$ $M_\infty = 0$	
Optimal allocation with smaller γ		$s_t, \ell_t \rightarrow 0$ $g_c < g_h, g_h = \bar{g}$ $\delta_t \rightarrow \delta^* > 0$ $M_\infty = 0$	$s_t, \ell_t \rightarrow 0$ $g_c = 0, c_t \rightarrow c^*$ $g_h = \bar{g}$
Optimal allocation with large γ	$s_t, \ell_t \rightarrow 0$ $g_c < g_h, g_h = \bar{g}$ $\delta_t \rightarrow 0$ $M_\infty > 0$	$s_t, \ell_t \rightarrow 0$ $g_c < g_h, g_h = \bar{g}$ $\delta_t \rightarrow 0$ $M_\infty > 0$	$\delta_t \rightarrow \infty$ $M_\infty = 0$
Existential risk in rule of thumb vs. optimal allocation	δ exponentially decays under rule of thumb. Optimal allocation changes pace of decay.	δ explodes under rule of thumb. Optimal allocation can contain growth in δ and achieves positive M_∞ .	Doomed to existential catastrophe whatever society does. Social planner may choose “end of growth.”

4 Evidence on Parameters

To understand our results, we have to know what realistic parameter values are and thus which world we live in. In particular, it would be very helpful to know whether $\epsilon > \beta$.

Evidence from broad trends in growth and existential risk

Over the past century, world economic output has grown manyfold. Technological risk to human civilization has arguably grown manyfold as well. Nuclear winter, catastrophic climate change, and genetically-engineered pandemics are all risks that have emerged in the past century. The Bulletin of Atomic Scientists, who publishes the “Doomsday Clock” assessing the likelihood of existential catastrophe, puts it as follows:

Our species has never before in its 200,000-year history been so close to a disaster as we are this century. It's unsettling enough that the Doomsday Clock has been set to an ominous 3 minutes to midnight (or doom) since 2015 [Note: 2 minutes to midnight since 2018]. But the real gravity of our situation only comes into focus once one realizes that before 1945, there was no need for the Doomsday Clock in the first place, given the low probability of doom. (Torres, 2016)

The key question then is what has happened to the fraction of safety spending as fraction of total output. If safety spending has not decreased as a fraction of total output, the functional form $\delta_t = \bar{\delta} C_t^\epsilon H_t^\beta$ immediately implies $\epsilon > \beta$.

Regrettably, I have not been able to find good data on safety spending as it is defined in this model. However, it seems like the effort spent on mitigating existential risk was approximately none a century or two ago. For example, nuclear disarmament and security, climate change abatement, and research on AI safety are all efforts that began only in the past century, and these efforts appear to be intensifying in the past decades. In that sense, it appears that the fraction of output spent on safety has increased.

It would clearly be desirable to collect better data on both the level of existential risk and the fraction of output spent on safety. Nevertheless, the general trend appears to imply $\epsilon > \beta$.

Evidence on $\frac{\alpha\lambda}{1-\phi}$

Jones and Romer (2010) give a broad and plausible range of $\frac{\alpha\lambda}{1-\phi} \in [1/2, 2]$. Note that this figure is for the economy as a whole. The R&D production function for the safety sector may be different, although we have no reason to believe it is, and so our base case should be that the nature of R&D is similar in both sectors. As such, I have imposed the same parameters on both the consumption and safety ideas production function in this paper.

Moreover, recent research by Bloom et al. (2017) demonstrates relatively sharply diminishing returns to research across a wide range of sectors using micro-evidence, indicating a low $\frac{\alpha\lambda}{1-\phi}$.

Evidence on γ

Most of the large empirical literature on the coefficient of relative risk aversion suggests $\gamma > 1$ is the relevant case. See e.g. Lucas (1994) on asset pricing and Chetty (2006) on labor supply. γ also traditionally equals the inverse of intertemporal substitution. The traditional evidence here suggests values well below one, implying γ well above 1; see Hall (2009) for a survey.

Evidence on ρ

Financial data reflecting consumer behavior tends to find pure time preferences in the range of 2%–5% (Pindyck, 2013). Weitzman (2007) finds data roughly consistent with a ρ of 2%. Nordhaus uses a rate of pure time preference of 1.5% in his seminal DICE climate change model (Nordhaus and Sztorc, 2013).

Implications

Our sparse evidence on the parameter values indicates that $\epsilon < \beta$ is unlikely, and the $\epsilon > \beta$ world may well be the one we live in. At the same time, the $\epsilon \gg \beta$ case appears surprisingly possible. If we take the high-end estimate of $\frac{\alpha\lambda}{1-\phi} = 2$, then an ϵ of $3/2$ and a β of $1/2$ would mean that $\epsilon \gg \beta$. If we look at the lower-end estimate of $\frac{\alpha\lambda}{1-\phi} = 1/2$, which may be the more realistic number given recent evidence on the sharply diminishing returns to research, even e.g. $\epsilon = 3/4$ and $\beta < 1/2$, or $\epsilon = 1/2$ and $\beta < 1/3$, would suffice for $\epsilon \gg \beta$. Perhaps our efforts at mitigating existential risk do not matter much after all—not because existential risk isn’t a problem, but because existential catastrophe is inevitable whatever we do.

It would clearly be beneficial to get better empirical evidence on these parameters. However, from now on, I will focus on the $\epsilon > \beta$ and $\epsilon \gg \beta$ case. This appears to be the empirically likely case.

Perhaps more importantly though, this is the case in which the effect of growth on risk is in doubt *and* this effect matters.

If $\epsilon < \beta$, (faster) growth straightforwardly reduces risk.

If $\epsilon \gg \beta$, growth does increase risk, but in some sense it doesn’t matter much: an existential catastrophe will wipe out humanity soon anyway. In the face of parameter uncertainty, the vast majority of the value of the future will be in cases where $\epsilon \gg \beta$, i.e. in cases where $M_\infty > 0$ and thus “astronomical value” is possible.

If $\epsilon > \beta$ (but $\epsilon \not\gg \beta$), growth does not straightforwardly reduce risk—without reallocation from consumption to safety, proportional growth in fact increases risk. At the same time, it is possible for $M_\infty > 0$ and thus to attain “astronomical value.”

In addition, I will focus on the case where $\gamma > 1$. This appears to be the empirically relevant case. Moreover, when $\epsilon > \beta$ and $\gamma \leq 1$, we again get $M_\infty = 0$ regardless of any intervention.

5 Transition Dynamics

The analysis so far has shed light on the long-run behavior of growth and risk. However, we live in a world far away from this asymptotic result. To understand the relationship between growth and risk as it might apply to today, I consider the transition dynamics of the (impatient) optimal allocation.

In particular, I analyze the case where $\gamma > 1$ and $\epsilon > \beta$.

5.1 Laws of Motion in the Optimal Allocation

The transition dynamics of the optimal allocation can be studied as a system of six differential equations in six “state-like” variables: s_t , ℓ_t , σ_t , δ_t , $y_t \equiv g_{At}$, and $z_t \equiv g_{Bt}$. With the addition of N_t , these variables then characterize all other variables. They each converge to constant values: $s^* = 0$, $\ell^* = 0$, $\sigma^* = \frac{\lambda\alpha g_B}{\rho + (\gamma-1)g_c + (1-\phi + \lambda\alpha)g_B}$, $\delta^* = 0$, $y^* = g_A$, and $z^* = g_B$.

Let \hat{s} denote the growth rate of s , $\hat{\ell}$ denote the growth rate of ℓ , and so on.

Proposition 12. *Laws of Motion in the Optimal Allocation*

In the optimal allocation, our “state-like” variables s_t , ℓ_t , σ_t , δ_t , y_t , and z_t

grow according to following laws of motion:

$$\hat{s} = \alpha z \frac{\lambda}{1-\lambda} (1-\ell) \frac{1-\sigma}{\sigma} - \alpha y \frac{\lambda}{1-\lambda} \frac{1-s}{s} \ell \frac{1-\sigma}{\sigma}, \quad (87)$$

$$\hat{\ell} = \frac{\theta_\ell (\mathbb{A} + \omega_\ell \theta_\sigma \mathbb{B})}{1 - \omega_\ell \omega_\sigma \theta_\sigma}, \quad (88)$$

$$\hat{\sigma} = \theta_\sigma (\mathbb{B} + \omega_\sigma \hat{\ell}), \quad (89)$$

$$\hat{\delta} = (\epsilon - \beta) \left(\bar{n} - \hat{\sigma} \frac{\sigma}{1-\sigma} \right) + \alpha (\epsilon y - \beta z) + \hat{\ell} \left(\epsilon + \beta \frac{\ell}{1-\ell} \right), \quad (90)$$

$$\hat{y} = \lambda (\bar{n} + \hat{s} + \hat{\sigma}) - (1 - \phi) y, \quad (91)$$

$$\hat{z} = \lambda \left(\bar{n} - \hat{s} \frac{s}{1-s} + \hat{\sigma} \right) - (1 - \phi) z, \quad (92)$$

where the following definitions have been used:

$$\omega_\ell = (\gamma - 1 + \epsilon - \beta) \frac{\sigma}{1-\sigma}, \quad (93)$$

$$\omega_\sigma = - \left(\frac{\ell}{1-\ell} (1 + \beta) + \epsilon \right), \quad (94)$$

$$\theta_\ell = \frac{(1-\ell) \left(1 + \frac{1-\ell}{\ell} \frac{\epsilon}{\beta} \right)}{1 + (\gamma - 1 + \epsilon + \beta \frac{\ell}{1-\ell}) (1-\ell) \left(1 + \frac{1-\ell}{\ell} \frac{\epsilon}{\beta} \right)}, \quad (95)$$

$$\theta_\sigma = \frac{1-\sigma}{1 + (\beta - \epsilon) \sigma - \lambda (1-\sigma)}, \quad (96)$$

$$\mathbb{A} = (\beta - \epsilon) \bar{n} + (1 - \gamma - \epsilon) \alpha y + \alpha \beta z - \rho - \delta + \frac{u(c_t)}{v_t}, \quad (97)$$

$$\mathbb{B} = (1 - \lambda) \frac{s}{1-s} \hat{s} + (\lambda + \beta - \epsilon) \bar{n} + \alpha \beta z - \alpha \epsilon y + \frac{u(c_t)}{v_t} - \alpha \lambda z \frac{1-\ell}{1-s_t} \frac{1-\sigma_t}{\sigma_t}, \quad (98)$$

and

$$\frac{u(c_t)}{v_t} = \frac{\ell}{1-\ell} \beta \delta \tilde{u} + \epsilon \delta \tilde{u}, \quad \tilde{u} = \bar{u} c_t^{\gamma-1} + \frac{1}{1-\gamma}, \quad (99)$$

$$c_t = \frac{\left(\frac{\delta}{\delta} \left(\frac{\ell}{1-\ell} \left(\left(\frac{z}{y} \right)^{\frac{1}{1-\phi}} \left(\frac{s}{1-s} \right)^{\frac{\lambda}{1-\phi}} \right)^\alpha \right)^{-\beta} \right)^{\frac{1}{\epsilon-\beta}}}{N_t}. \quad (100)$$

Proof. See Appendix A.13. \square

5.2 Numerical Simulation

Simulating this system of equations yields a candidate transition path for each set of parameters. These candidate transition paths feature two broad dynamics that emerge for different combinations of parameter values. The first dynamic features growth rates of A and B (and thus c and h) that start very high (with c very close to 0) and then fall to the steady state. The second dynamic features growth rates of A and B (and thus c and h) that start small and then rise over time to the steady state. In trying to understand the long-term dynamics of our civilization, the latter appears to be the relevant case. Over the period of recorded history, consumption was initially broadly flat (but nonzero). Then, growth sped up. Thus, I will focus on the second case. Although the exact dynamics depend on the specific parameter values of course, the example below illustrates the central qualitative features of this case.

I set $\gamma = 1.5$, $\epsilon = 0.4$, $\beta = 0.3$, $\rho = 0.02$, and $\bar{n} = 1\%$. These are meant to be reasonable values for illustration; other values produce similar results. I choose the other parameter values, including ϕ , λ , $\bar{\delta}$, and \bar{u} , to target several stylized facts about the world. In particular, I seek to find a year t_0 with a value of life-year as a ratio to per capita consumption (\tilde{u}) of 4 (corresponding e.g. to per capita consumption of \$20,000 and a value of a life-year of \$80,000), in which consumption per capita grows at around 1 percent per year, around 95% of workers are in the consumption sector, and the hazard rate is approximately 0.1%. This 0.1% rate of existential catastrophe has become a relatively widely used benchmark; see Stern (2006) and Méjean et al. (2017, 2019).

This is not meant to be a formal calibration in any sense: we do not have good information about many of the parameters. This exercise is merely meant to illustrate the qualitative dynamics; the calibration helps us use a reasonable set of parameters. Critically, note that the qualitative dynamics of these results are similar for other parameter choices, such as different γ , different ϵ and β , and different ρ . I explain the details of the simulation in Appendix B.1.

Figure 2 shows the key allocation of workers and scientists to consumption along the transition path. Figure 3 shows the growth rates of consumption and safety along the transition path. Figure 4 shows the hazard rate δ along the transition path.

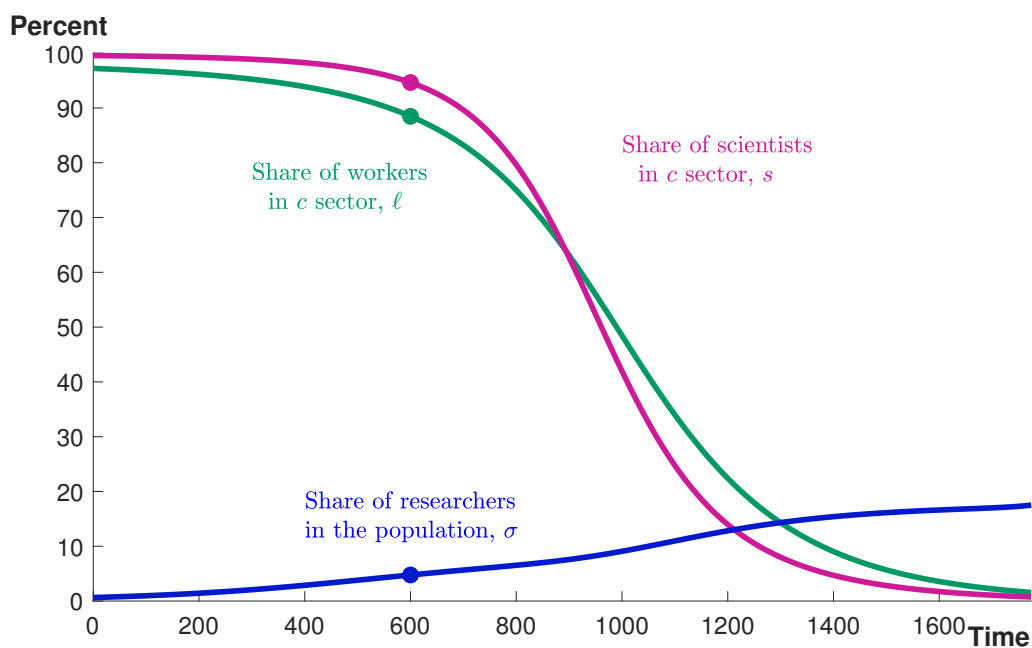


Figure 2: The allocation along the transition path. Time 600 corresponds to today and the values at this date are highlighted in the graph. A period represents a year.

Consider first the allocation variables displayed in Figure 2. At the time representing today, nearly all scientists and workers are in the consumption sector. As consumption grows and thus the relative value of life \tilde{u} grows, both s and ℓ decline as resources are shifted to the safety sector. Note that initially, safety is increased by shifting workers towards the safety sector; only later are scientists shifted towards the safety sector. Both s and ℓ eventually settle in to their asymptotic, exponential decline to zero. The share of scientists in the population σ rises steadily to its steady-state value.

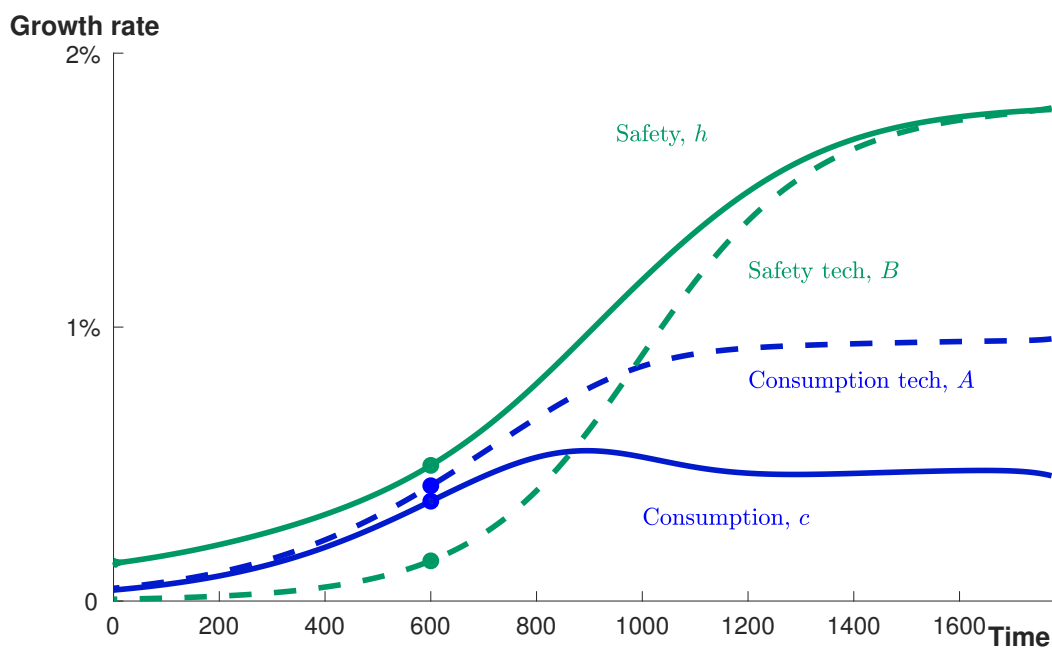


Figure 3: The growth rates along the transition path. Time 600 corresponds to today and the values at this date are highlighted in the graph. A period represents a year.

Next, consider the growth rates along the transition path in Figure 3. The growth rates of consumption technology A and thereby consumption per capita c rise steadily, accelerating from a low initial level to higher consumption growth at the time representing today. We saw that as consumption grows and the value of life rises, workers and scientists are shifted to the safety sector. This causes the growth rate of A to level off and consumption growth to slow, while the growth of safety per capita h accelerates. Note that the additional safety growth is driven by shifting workers to the safety sector at first; only after a while does the growth of safety technology B begin to accelerate. All growth rates eventually converge to their constant asymptotic values, with consumption growing significantly slower than safety. However, consumption does continue growing at a constant exponential rate.

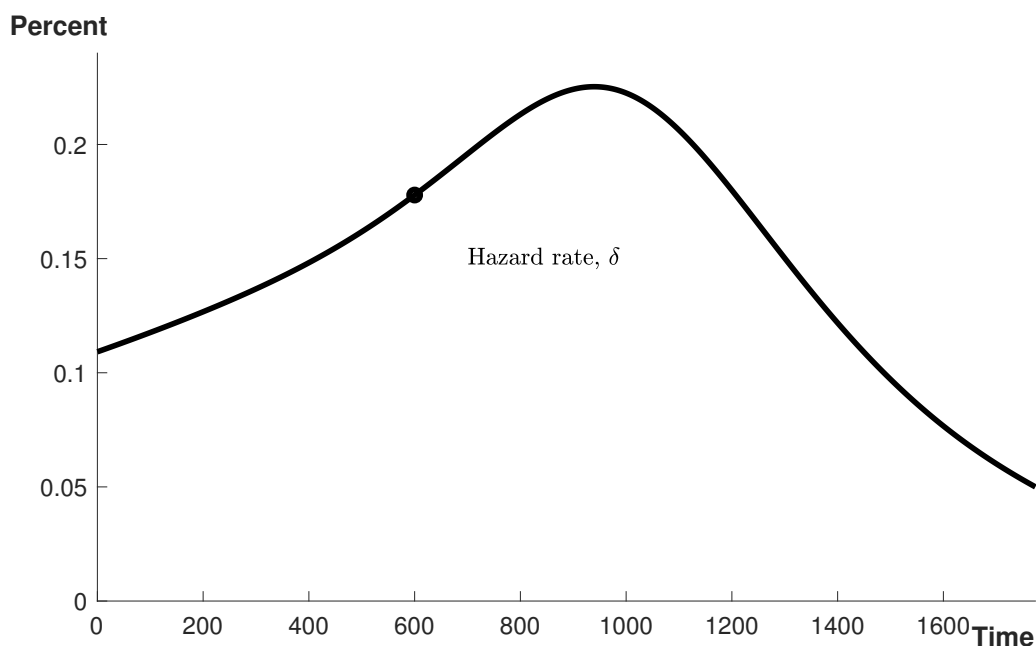


Figure 4: The hazard rate along the transition path. Time 600 corresponds to today and the value at this date is highlighted in the graph. A period represents a year.

Finally, consider perhaps the most interesting dynamic: the hazard rate along the transition path in Figure 4. The key qualitative dynamic is that the hazard rate curve has an inverted U-shape. The hazard rate starts at a relatively low level. Yet since $\epsilon > \beta$, existential risk grows with scale, so δ grows. This means that at the time representing today, the risk of an existential catastrophe is much higher than it was hundreds of years ago. As consumption grows, the value of life rises and so resources are shifted to safety. This slows the growth rate of δ , yet existential risk keeps rising—the scale effect still dominates for a while. Eventually, the growth in safety relative to consumption outpaces the scale effect, so the δ -curve bends: existential risk starts to fall. The hazard rate δ ultimately decays exponentially.

Recall that what matters in determining the long-term probability of humanity’s survival is the area under the hazard rate, since $M_\infty = e^{-\int_0^\infty \delta_s ds}$. The exponential decay of the hazard rate on the asymptotic path ensures that $\int_0^\infty \delta_s ds$ is finite, and so $M_\infty > 0$. Extrapolating from the simulation, the long-run probability of human civilization’s survival conditional on surviving

to the time that represents today is approximately 19.3%. However, note that I calibrated the above simulation to have a δ of approximately 0.1% today; a different calibration would rescale this curve and thus change the magnitude of the survival probability.

5.3 Discussion: The Existential Risk Kuznets Curve

This inverted U-shape of the hazard rate curve is related to the literature on the “environmental Kuznets curve,” which posits an inverted U-shape relationship between economic development and pollution (see Brock and Taylor (2005) for an overview). The mechanism at work in this model is similar to the classic Stokey (1998) paper on the theory behind the environmental Kuznets curve: if $\gamma > 1$, richer societies care less about increasing consumption and more about other things, such as the environment, or, in this case, life. Initially, pollution rises with scale, but eventually declines as the relative value of environmental protection increases, producing a hump-shaped pollution curve. While the matter at hand is very different—environmental degradation that additively reduces utility versus existential catastrophes that imperil human civilization—the analogy supports the soundness of the result.

There are two important things to keep in mind, however, about what we might call the “existential risk Kuznets curve.” First, the timescales involved here appear to be very long, involving hundreds or even thousands of years of economic development. Zooming in even a few hundred years around the present in the graph above, we would likely only increasing risk, much as some argue we have seen in the past century. On the one hand, this shows the value of economic theory: it allows us to gain a long-run perspective on potential societal dynamics. On the other hand, this means we cannot easily test this model prediction empirically, giving us reason for caution.

Secondly, note that this existential risk Kuznets curve appears in the transition dynamics of the *optimal* allocation. Considering that existential risk mitigation is a global public good, it is unlikely resources are allocated to safety optimally in the real world. As such, this should not be taken to be a prediction of what a particular country with a particular set of institutions will do with regard to existential risk.

Nevertheless, there are a number of reasons why we might still be interested in the transition dynamics under the (impatient) optimal allocation. For one, since there are very long timescales involved here, it is very hard to

know (and thus model) what government and societal institutions will evolve to deal with existential risk. However, the ideal these institutions will likely aim at is the optimal allocation. The optimal allocation might thus be a rough proxy for the real-world allocation.

Moreover, the (impatient) optimal allocation represents what I would call the “democratic possibilities frontier” or the “impatient public possibilities frontier.” Those who are principally concerned about the long-run future of humanity and advocate for a zero rate of pure time preference might want us to spend as much as possible on safety in order to avoid existential catastrophe and enable human flourishing millions of years into the future. Indeed, even in the Hamiltonian of the optimal allocation, the relative value of life \tilde{v}_t is a discounted term; the lower your discount rate ρ , the more you would want to spend on safety. However, the broader public is not so patient. As the empirical evidence cited earlier shows, people tend to have a (relatively large) positive rate of pure time preference; the public is impatient. Even perfectly designed institutions that take into account existential risk externalities will ultimately be constrained by the degree to which society actually cares about the future—they will be constrained by an impatient public. The existential risk Kuznets curve illustrates the implications of this impatience. On the one hand, this impatience results in a period of initially rising levels of risk. For example, this might mean that the arguably rising level of existential risk of the past century is not necessarily a market failure, but may well be part of the optimal path given positive pure time preference. On the other hand, rising standards of living lead even the most impatient public to start caring more about safety and averting an existential catastrophe. This leads workers and scientists to be shifted to the safety sector, eventually causing the hazard rate δ to exponentially decline. Even if people are impatient, if you make them well off enough, they will start caring about existential risk.

Seeing the arguably rising levels of existential risk in the past century, some might call for an end to economic growth. Yet this existential risk Kuznets curve indicates that stopping economic growth would be deleterious: it would simply freeze the hazard rate at a high level, leading to a fatal catastrophe sooner or later. Economic growth enables even an impatient public with a high rate of pure time preference to start caring about life, thus ultimately reducing risk and even leading to positive M_∞ .

Some prominent thinkers have previously posited that humanity is passing through a unique period with an elevated risk of technological catastro-

phe. Sagan (1994) calls this the “time of perils.” Parfit (2011, p. 616), concurs:

We live during the hinge of history. Given the scientific and technological discoveries of the last two centuries, the world has never changed as fast. We shall soon have even greater powers to transform, not only our surroundings, but ourselves and our successors. If we act wisely in the next few centuries, humanity will survive its most dangerous and decisive period. Our descendants could, if necessary, go elsewhere, spreading through this galaxy.

This existential risk Kuznets curve provides theoretical evidence that grounds the intuition that we are living in a “time of perils.” *We may be economically advanced enough to have created the means for our permanent destruction, but not economically advanced enough to care enough about decreasing this existential risk.*

This “time of perils” has profound implications. For instance, those alive today who care about preserving the long-term future of humanity may have extraordinary altruistic leverage. By working to reduce existential risk now (increasing the resources dedicated to safety), they can reduce the area under the “hump” of the hazard rate δ . This in turn increases M_∞ , unlocking tremendous value. Moreover, since so few resources are dedicated to safety at the moment, there are likely very high marginal value opportunities available to work on safety. This is a unique situation. Suppose existential risk did not decline to zero exponentially: then $M_\infty = 0$ regardless—the existential risk curve would never bend—so reducing risk now would not change the probability of a long and flourishing future of humanity. And if existential risk did not initially increase, it would never be such a substantial challenge and there wouldn’t be such high marginal value opportunities to work on reducing it.

6 Does Faster Growth Increase the Probability of Existential Catastrophe?

Faster economic growth is conventionally seen as a great boon for humanity. Yet when considering existential risk, this picture becomes more muddled.

Faster economic growth might speed up the development of potentially dangerous technology, such as powerful AI, or accelerate the pace of climate change. What if faster economic growth—in a world that does not (yet) value life highly—also accelerates the growth in risk? Could the side effect of mundane efforts to e.g. make trade more efficient or increase labor force participation be increasing the probability of an existential catastrophe? While the existential risk Kuznets curve explicated in the last section suggests we should at least want some economic growth even from the perspective of maximizing M_∞ , this does not tell us anything about how the pace of economic growth affects the probability of an existential catastrophe.

First, consider a generic, uniform shock—e.g. more people working—on the balanced growth path of the rule of thumb allocation. Since the fraction of workers and scientists working in the safety sector is fixed, this increases the number of scientists and workers in the safety sector and consumption sector by the same proportion. If $\epsilon < \beta$, this shock therefore decreases the hazard rate δ . If $\epsilon = \beta$, there is no effect on δ . But if $\epsilon > \beta$, the shock increases the hazard rate δ because of the scale effect.

When $\epsilon < \beta$, faster growth reduces existential risk even in the rule of thumb allocation. Yet when $\epsilon > \beta$, accelerating growth also accelerates the growth in risk if the allocation of resources to safety does not adjust.

I will look at what happens when we accelerate growth in the (impatient) optimal allocation. In particular, I will look at the $\epsilon > \beta$ case, since in the $\epsilon < \beta$ case, faster growth reduces risk even when the allocation of resources to safety does not adjust. As explained earlier, although the real-world allocation may be imperfect, the optimal allocation might be a rough proxy for how societies will decide to allocate resources to safety in the long run. Moreover, the optimal allocation represents the “democratic possibilities frontier”: the (high) positive rate of pure time preference the public appears to have dictates the degree to which societies can trade off consumption for safety. I also focus on the $\gamma > 1$ case as in the previous section.⁶

6.1 Simulating an Acceleration of Growth

First, consider what happens when growth is faster for a given time period, resulting in permanently higher economic output (i.e. this results in a per-

⁶Note that if $\gamma \leq 1$, accelerating growth would not matter for the chances of human civilization’s survival in the long run: $M_\infty = 0$ anyway, regardless of whether growth is faster or slower.

manent level effect).

In this model, population growth is the driving force behind economic growth. More population means more nonrival ideas which means more output. Moreover, it is easy to manipulate population growth in our model by manipulating \bar{n} . Thus, I consider the effect of accelerated population growth. In particular, I simulate 2% (instead of 1%) population growth for 30 years around the time representing today. We can take this to literally represent the effect of some pro-natalist policy. However, the basic dynamics illustrated below should apply to a broad class of generic accelerations in growth, e.g. increasing labor force participation, increasing human capital, increasing the number of “effective” people by making global exchange easier, or increasing research effort.

See Appendix B.2 for details of how I simulate the acceleration in growth.

The following figures compare the transition path with steady growth and the transition path with a period of accelerated growth. The transition path with steady growth is depicted with the solid colors; the transition path with a period of accelerated growth is depicted with the lighter colors. Figure 5 shows the growth rates of consumption and safety along the transition path. Figure 6 shows the fraction of workers and scientists allocated to consumption along the transition path. Figure 7 shows the relative value of life \tilde{u} along the transition path. Figure 8 shows the hazard rate δ along the transition path.

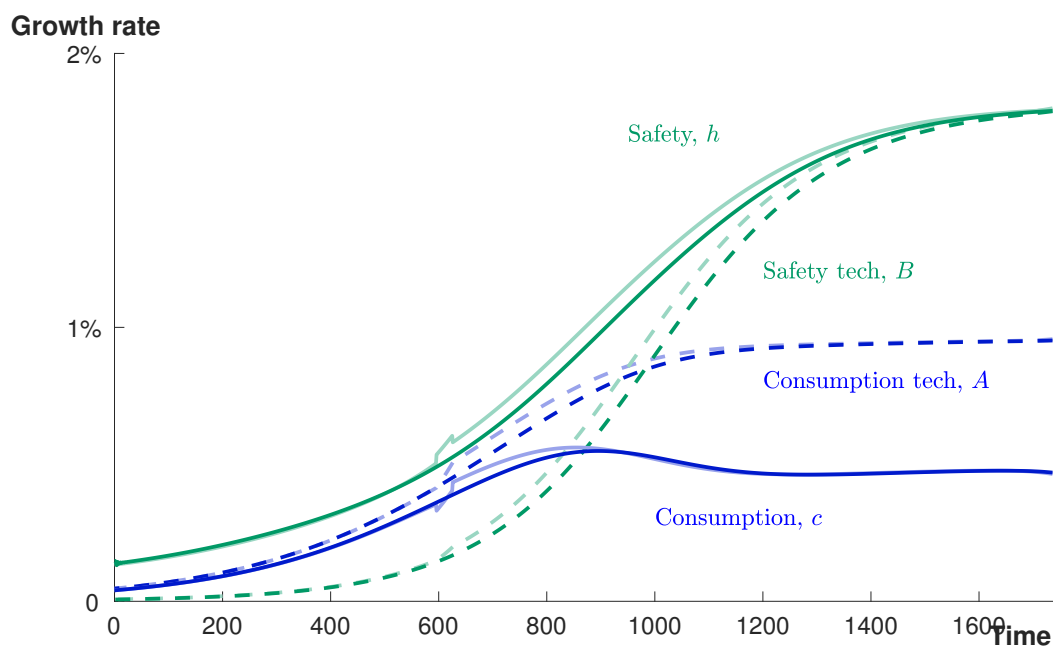


Figure 5: The growth rates along the transition path, comparing steady growth (solid colors) and a period of accelerated growth (lighter colors). A period represents a year.

Consider first the growth rates depicted in Figure 5. At around time 600, population growth accelerates for 30 years. This accelerates the growth rates of consumption technology A and safety technology B on the transition path with the accelerated growth. Both growth rates remain higher for a while, until they eventually converge to the same steady state as along the transition path with steady growth. The higher growth rates of A and B increase the growth rates of c and h . g_c and g_h are thus higher on the transition path with a period of accelerated growth, until these too converge to the same steady state as along the transition path with steady growth. Note that consumption growth actually initially decelerates a bit during the period of accelerated population growth to compensate for the scale effect of faster population growth.

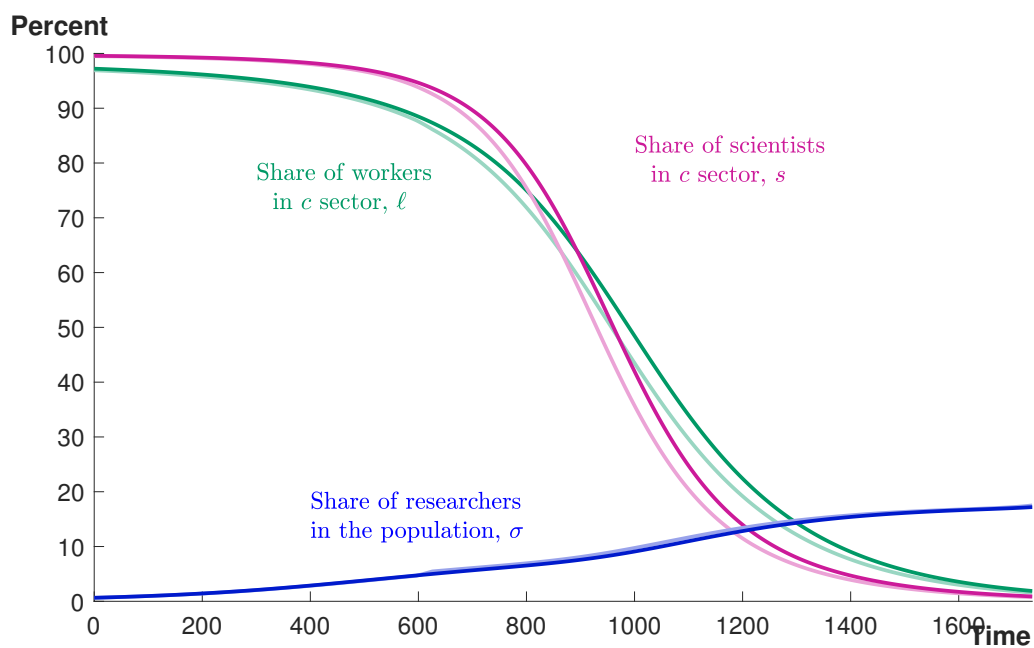


Figure 6: The allocation along the transition path, comparing steady growth (solid colors) and a period of accelerated growth (lighter colors). A period represents a year.

Next, consider the key allocation variables shown in Figure 6. The growth of the share of researchers in the population slightly increases during the period of accelerated growth. More importantly, along the transition path with the period of accelerated growth, workers and scientists are shifted to safety earlier than along the transition path with steady growth.

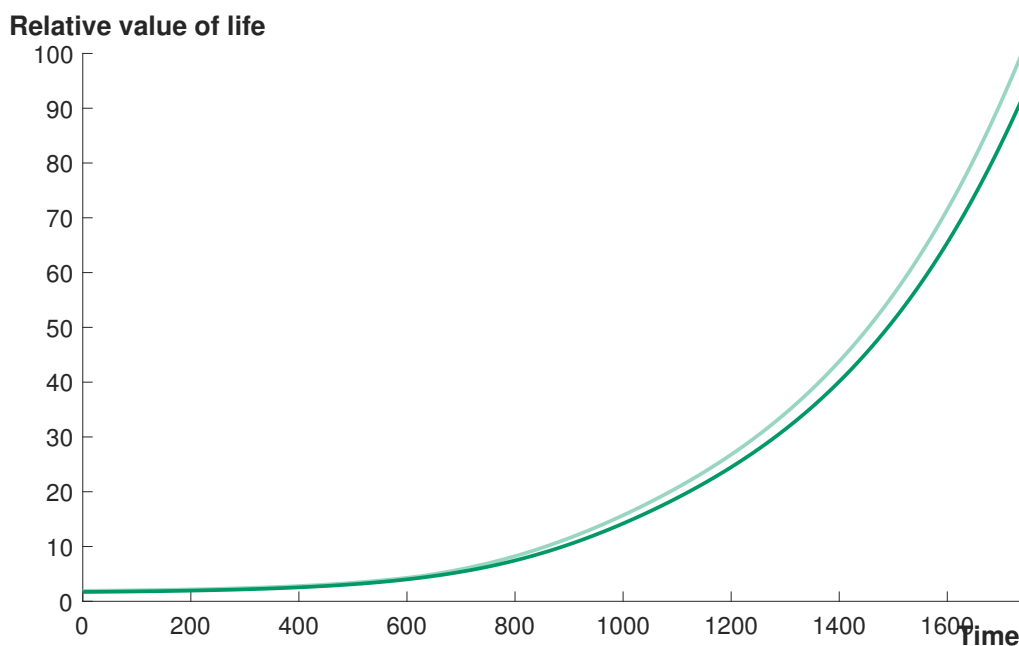


Figure 7: The relative value of life \tilde{u} along the transition path, comparing steady growth (solid green) and a period of accelerated growth (light green). A period represents a year.

To understand the dynamics at play, consider Figure 7, which compares the relative value of life \tilde{u} along the transition path with accelerated and steady growth. At approximately time 600, when there is a period of faster population growth, \tilde{u} begins to diverge along the two transition paths. After time 600, \tilde{u} is higher along the transition path with accelerated growth compared to along the transition path with steady growth. Recall the growth rates illustrated in Figure 5: the acceleration of growth meant faster consumption growth. Faster consumption growth in turn means that along the transition path with accelerated growth, people are richer, earlier, than they would have been with steady growth. Since $\tilde{u} = \frac{u(c)}{u'(c)c} = \bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma}$ and $\gamma > 1$, these richer people then value life more highly; they are more concerned for safety, earlier. Thus, resources are shifted to safety earlier, as we saw in the allocation dynamics.

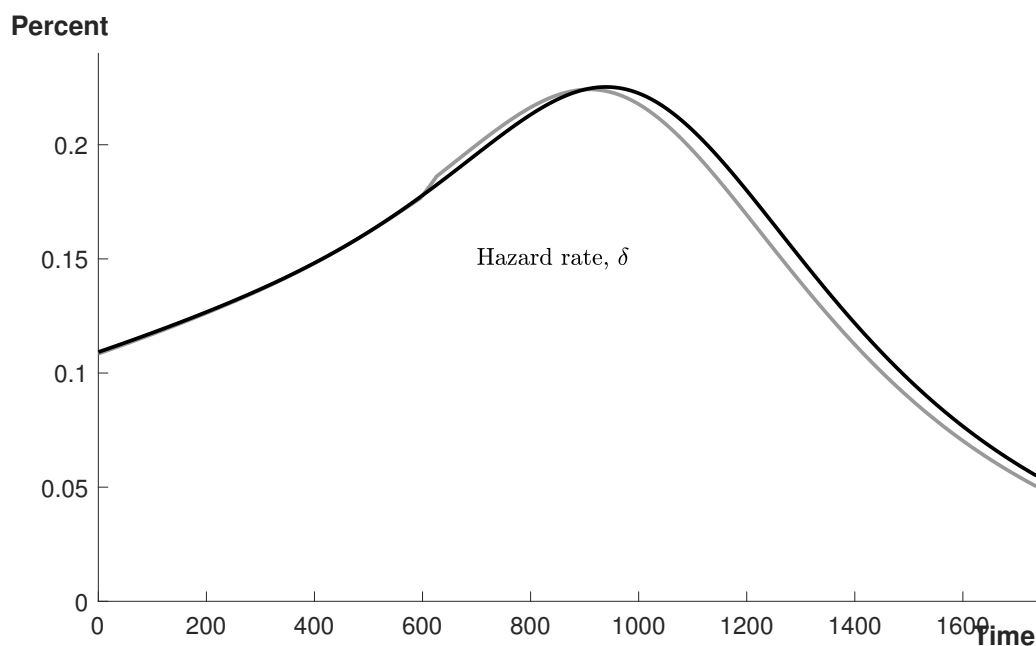


Figure 8: The hazard rate along the transition path, comparing steady growth (black) and a period of accelerated growth (gray). A period represents a year.

Consider the hazard rate δ depicted in Figure 8. After the period of accelerated growth around time 600, it initially seems as if all of the worries about faster economic growth have been confirmed: the acceleration in growth also accelerates the growth in the hazard rate. This is all an observer at the time—or even hundreds of years later—would be able to observe. Armed with empirical data, this observer would conclude that faster growth increased existential risk.

Yet zooming out, this is not so. The acceleration of growth also accelerates the rise of the relative value of life \tilde{v} . As such, \tilde{v}_t is higher: people start caring more about safety earlier in the world with a period of accelerated growth compared to the world with steady growth. Resources are shifted to safety sooner, and thus the hazard rate curve bends earlier. In a sense, the period of faster growth accelerates the movement along the existential risk Kuznets curve. As a result, the overall area under the hazard rate curve is lower—and recall that this is all that matters for the long-run probability of civilization's survival.

As before, on the steady growth transition path, M_∞ conditional on surviving to the time that represents today is approximately 19.3%. However, on the transition path with a period of accelerated growth, M_∞ conditional on surviving to the time that represents today is approximately 20.8%. Thus, we see how the period of accelerated growth, despite increasing risk initially, improves the changes of humanity's survival in the long run! This effect is not trivial: faster growth for a relatively short period of time now appears to result in increasing the long-run probability of human survival by 1.5 percentage points. Instead of faster economic growth being a problem in the context of existential risk, this suggests that faster economic growth could actually contribute to the challenge of mitigating existential risk—even when people are impatient.

When previously discussing the existential risk Kuznets curve, I mentioned that we may well be living through a “time of perils.” This analysis suggests that one way to increase the probability of humanity's survival is to simply try to get through the “time of perils” as quickly as possible. This may counterintuitively mean accelerating the increase in existential risk initially (if we are currently on the upward-sloping part of the hazard rate curve). However, this accelerationist strategy would ultimately decrease the area under the hazard rate curve and increase the probability of a long, flourishing future.

The reverse of the above happens when growth decelerates: the movement along the existential risk Kuznets curve decelerates, and society is stuck with higher levels of existential risk for longer, in turn dramatically decreasing the long-run probability of humanity's survival. Slower growth—even just for a while—doesn't just mean lower living standards, but potentially a much higher chance of an existential catastrophe and a much lower chance of a long future of humanity. This should strike fear of even short-term stagnation into the hearts of all those who care about the long-term future.

The key condition here is that $\epsilon \not\gg \beta$. The acceleration of growth initially increases risk due to the scale effect—but since the scale effect of ideas is larger than the scale effect of existential risk, it was still possible to mitigate risk eventually once \tilde{u} got high enough and people started caring. Yet if $\epsilon \gg \beta$, the higher \tilde{u} does not matter: even if society wanted to mitigate the additional risk later on, it would be impossible.

6.2 Simulating a Transitory Boom

So far, we have been looking at an acceleration in growth that results in a permanent level effect. What happens when we have a transitory economic boom, i.e. a time of faster growth that doesn't change the long-run level of output? For the reasons stated before, we again manipulate population growth, letting it be 2% for 40 years and then 0% for the 40 years after that (instead of a steady 1%). Thus, the long-run population is unaffected; there is simply a temporary upward blip. We may interpret this literally as the effect of a transitory baby boom. However, the basic dynamics illustrated below should apply to a broader class of transitory booms, e.g. an economic boom as part of the business cycle in which the economy is operating over capacity.

See Appendix B.2 for details of how I am simulating the acceleration in growth.

The following figures compare the transition path with steady growth and the transition path with a transitory boom. The transition path with steady growth is depicted with the solid colors; the transition path with a transitory boom is depicted with the lighter colors. Figure 9 shows the growth rates of consumption and safety along the transition path. Figure 10 shows the key allocation of workers and scientists to consumption along the transition path. Figure 11 shows the hazard rate δ along the transition path.

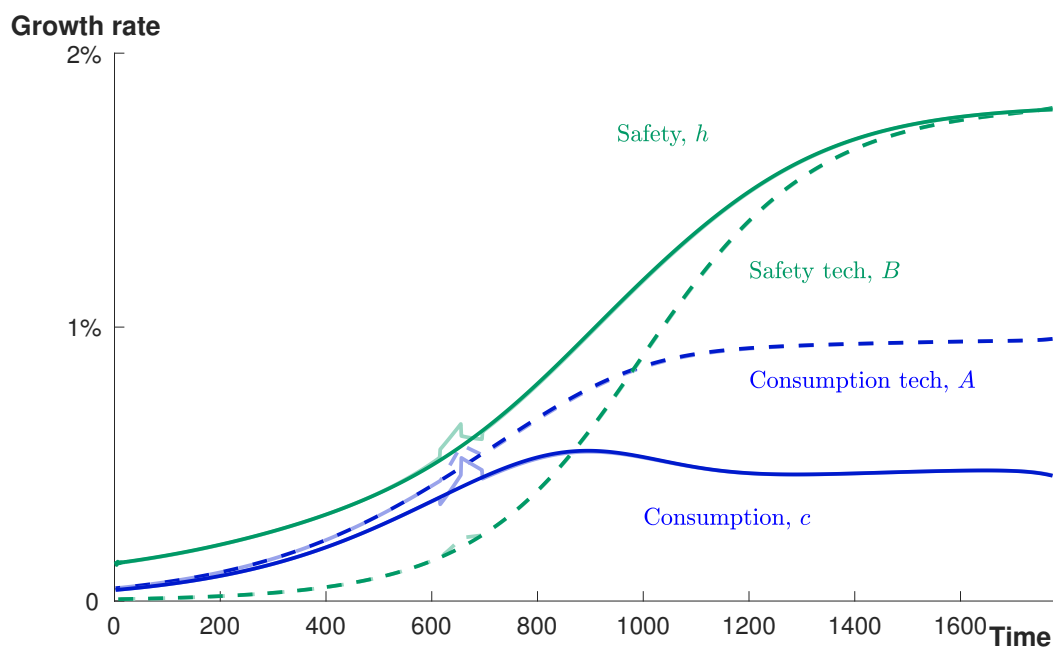


Figure 9: The growth rates along the transition path, comparing steady growth (solid colors) and a transitory boom (lighter colors). A period represents a year.

Consider first the growth rates shown in figure 9. The faster population growth initially accelerates the growth rates of both consumption technology A and safety technology B . Growth in both of these then slows down when population growth is slower during the time of slower population growth. The upward blip in the growth rates of A and B in turn lead to an upward blip in the growth rates of c and h . Nevertheless, after the temporary boom, all growth rates are the same, as had the boom not happened.

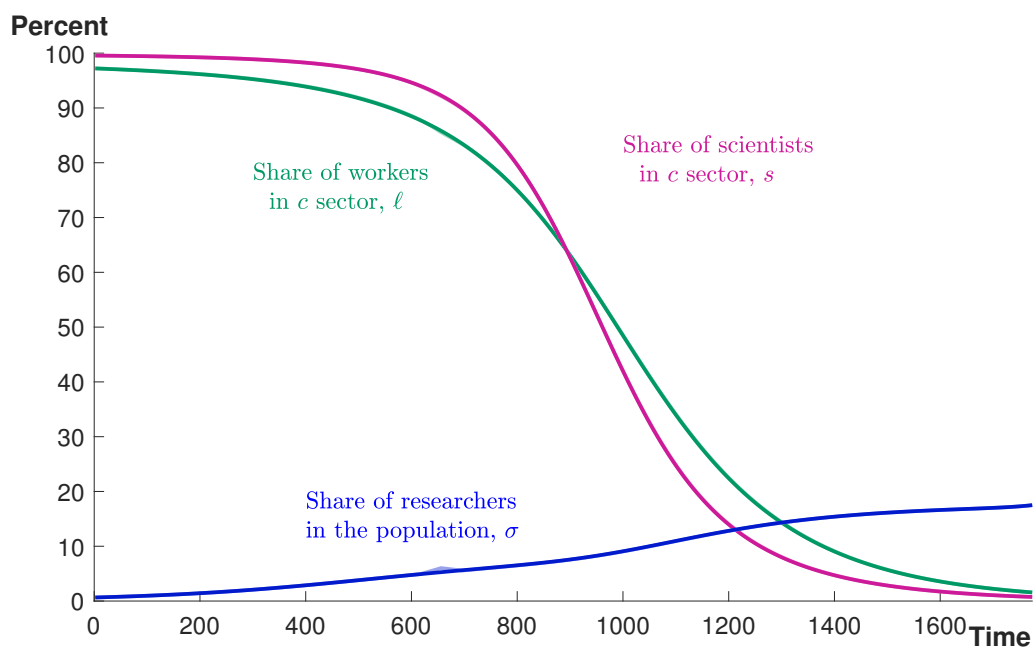


Figure 10: The allocation along the transition path, comparing steady growth (solid colors) and a transitory boom (lighter colors). A period represents a year.

Next, consider the key allocation variables depicted in figure 10. There is a temporary upward blip in the fraction of population working as researchers. There is also a temporary downward blip in the fraction of workers and researchers working in the consumption sector. Yet unlike when growth was accelerated resulting in a permanent level effect, this temporary economic boom does not change the long-term trajectory of the relative value of life; thus, the long-term trajectory of the allocation variables is unchanged.

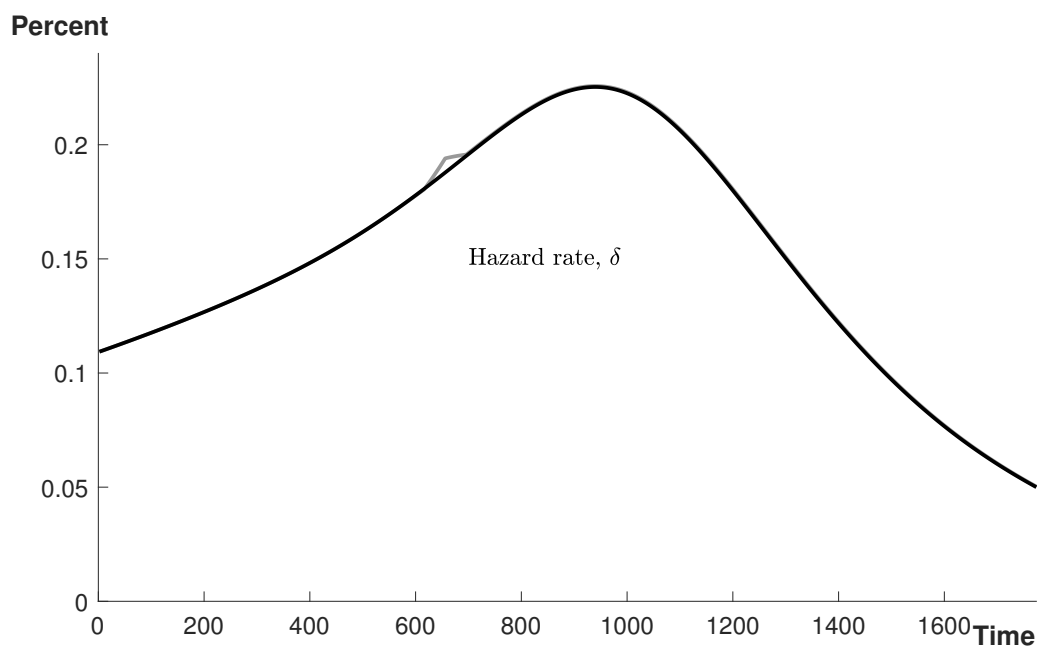


Figure 11: The hazard rate along the transition path, comparing steady growth (solid black) and a temporary boom (gray). A period represents a year.

Consider the hazard rate illustrated in figure 11. The long-run trajectory of the hazard rate is the same.

Nevertheless, this doesn't mean that temporary boom has no effect: there is an upward blip in the hazard rate during the boom. Recall again that what matters in determining the long-term probability of humanity's survival is the area under the hazard rate curve. This upward blip in the hazard curve increases the area under the hazard curve, which reduces humanity's long-term survival probability. Extrapolating from the simulation, conditional on surviving until the time representing today, the difference in long-term survival probabilities is approximately 0.17 percentage points. Considering this may just be the effect of e.g. the business cycle, and the outcome at stake is whether humanity goes extinct or not, this is again a surprisingly large effect.

The opposite occurs when we simulate a temporary bust, i.e. a slow-down in (population) growth followed by an increase in growth such that the long-term trend remains the same. Then, the hazard rate curve exhibits

a downward blip, which increases the long-term probability of humanity’s survival.

We previously saw that a period of accelerating growth can increase the long-term probability of humanity’s survival. Here, we thus add important nuance: the additional growth has to result in a permanent level effect. Simply “juicing” growth for a while may actually backfire, reducing the probability of humanity’s survival. Nevertheless, the intuition we developed remains the same: we want to get through the “time of perils” as quickly as possible. Stagnation—in this case the “cooling off” after a transitory boom—during the “time of perils” is deleterious.

6.3 Patience vs. Growth

The key mechanism at work in this paper is that growing consumption grows people’s relative value of life \tilde{u} (when $\gamma > 1$). The period of accelerated growth improves the chances of civilization’s survival in the long run because it accelerates the rise in the relative value of life \tilde{u} . As people grow richer, they care more about preventing an existential catastrophe and demand more safety.

By contrast, philosophers who are concerned about the long-term future often appeal to ethical arguments for a zero rate of pure time preference. They care about existential risk mitigation not because of a high \tilde{u} , but because of low or no utility discounting.

How do these two mechanisms—increasing consumption vs. reducing the rate of pure time preference ρ —compare in terms of increasing concern for safety?

Recall that in the optimal allocation,

$$\frac{1 - \ell_t}{\ell_t} = \frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t},$$

$$\frac{1 - s_t}{s_t} = \frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t} \cdot \frac{\rho - g_{pat} - \phi g_{At}}{\rho - g_{pat} - \phi g_{Bt}} \cdot \frac{g_{Bt}}{g_{At}}.$$

Both the allocation of workers to safety and the allocation of scientists to safety are proportional to \tilde{v}_t . \tilde{v}_t represents people’s demand for safety. Recall that

$$\tilde{v}_t = \frac{\tilde{u}_t}{\rho - \delta_t + g_{vt}}, \quad \tilde{u}_t = \frac{u(c_t)}{u'(c_t)c_t}.$$

We see that people's concern for life depends on both \tilde{u} , which in turn depends on consumption, and ρ .

Thus, we can compare how lowering ρ —making people more patient—and increasing c_t —making people better off and thus increasing \tilde{u} —compare in terms of increasing \tilde{v} , people's concern for life. Although this concern for life does not necessarily translate directly to the allocation of resources to safety in the real world as it does in the optimal allocation, we would hope that the real-world allocation responds to the people's demand for safety in the long run.⁷

Proposition 13. *Elasticities of Concern for Life*

Suppose the marginal utility of consumption falls rapidly, such that $\gamma > 1$. Let $E_\rho^{\tilde{v}}$ be the elasticity of \tilde{v} with respect to ρ . Let $E_c^{\tilde{v}}$ be the elasticity of \tilde{v} with respect to c_t . As $\tilde{u} \rightarrow \infty$ and $\delta \rightarrow 0$,

$$E_\rho^{\tilde{v}} \rightarrow -1, \tag{101}$$

$$E_c^{\tilde{v}} \rightarrow (\gamma - 1). \tag{102}$$

In particular, when \tilde{u} is large and δ is sufficiently smaller than ρ , $E_\rho^{\tilde{v}} \approx -1$ and $E_c^{\tilde{v}} \approx (\gamma - 1)$.

Proof. See Appendix A.14. □

For large enough \tilde{u} (i.e. people are already decently well off and care about life somewhat), the elasticity of \tilde{v} with respect to ρ is approximately -1 . Halving ρ roughly doubles the concern for life \tilde{v} . Moreover, the elasticity of \tilde{v} with respect to c is approximately $(\gamma - 1)$. For example, if $\gamma = 2$, doubling c roughly doubles the concern for life \tilde{v} . The larger γ , the larger this elasticity, since a larger γ means the marginal utility of consumption decreases more rapidly and so the relative value of life \tilde{u} increases more rapidly.

I have computed the approximate elasticities for different values of γ below. To help clarify the comparison, in the third column, I note what ρ would have to be reduced to, from a base of $\rho = 2\%$, to match the increase in the concern for life \tilde{v}_t from a doubling of consumption.

⁷Note that when I am referring to ρ , this ρ is the rate of pure time preference without regard for increasing population. In a total utilitarian setting, the rate of pure time preference is $\rho + \bar{n}$. Thus, the elasticities with regard to the rate of pure time preference in a total utilitarian setting would be lower if $\bar{n} > 0$.

Table 2: Patience vs. Growth: Comparison of Effect on Concern for Safety

γ	$E_c^{\tilde{v}}$	ρ equivalent to doubling consumption
1.1	0.1	1.87%
1.5	0.5	1.41%
2	1	1%
4	3	0.25%

For γ close to 1, even doubling consumption is equivalent only to a small change in pure time preference in terms of regard for safety. Increasing consumption is relatively ineffective in increasing people’s concern for safety. Yet for larger γ , increasing consumption has very large effect on the concern for safety—equivalent to very large reductions in pure time preference.

Note that I have been using an approximation for the elasticities that is valid for sufficiently large \tilde{u} . For lower \tilde{u} , g_{vt} is higher, so $E_\rho^{\tilde{v}}$ is lower. At the same time, for lower \tilde{u} , $E_c^{\tilde{v}}$ is higher—in fact, $E_c^{\tilde{v}} \rightarrow \infty$ as $\tilde{u} \rightarrow 0$. This indicates that the above numbers are a lower bound for the relative effectiveness of growing c versus lowering ρ . If people are poorer and \tilde{u} is lower, increasing c is much more effective relative to decreasing ρ in increasing the concern for safety than the numbers above imply.

Nevertheless, the general takeaway is clear. Making people better off could increase concern for safety and thus demand for existential risk mitigation in a way that would be equivalent to significant changes in people’s attitude toward the future.

7 Conclusion

Human activity can create or mitigate existential risks. Analyzing this in a model of endogenous growth, when the scale effect of existential risk is moderate and the marginal utility of consumption declines quickly enough, this paper grounds the intuition of some prominent thinkers that humanity may be in a critical “time of perils.” We may be economically advanced enough to be able to destroy ourselves, but not economically advanced enough that we care about this existential risk and spend on safety. This “time of perils” implies that working on reducing existential risk now could be very impactful from an altruistic perspective.

Faster economic growth appears to generally improve the odds of humanity's survival. When the world is not very fragile, growth straightforwardly reduces risk. When the world is extremely fragile, growth increases risk, but humanity is doomed to destruction anyway—the odds of humanity's survival are zero anyway. In the moderate fragility case, faster economic growth, while initially increasing risk, can help us get through this “time of perils” more quickly and thus increases the long-run probability of humanity's survival. In particular, note that this effect doesn't require a permanently higher growth rate; the improvement in humanity's odds of survival are potentially substantial even for a temporary period of faster growth. Conversely, even a temporary period of slow economic growth—as we have arguably been experiencing in the developed economies in recent decades—could substantially curtail the future of human civilization.

This model suggests even if you care only about the very long-term future of humanity, the pace of economic growth in the short run could be key to whether we make it there.

This paper suggests many future research directions. It would clearly be desirable to get better empirical data on the scale effect of existential risk. More broadly, a better understanding of how exactly existential risk is created and mitigated—and therefore modeled—would be helpful. It would also be interesting to look at the implications of a decentralized allocation, as well as possible mechanisms to efficiently provide for the global public good of existential risk mitigation. Finally, from the perspective of maximizing altruistic impact, it would be valuable to compare the impact on the long-run probability of humanity's survival from working on policies that might accelerate the rate of growth to direct work on reducing existential risk by funding the safety sector.

Appendices

A Proofs and Derivations

A.1 Proof of Proposition 1

Note that

$$g_{At} = \frac{\dot{A}_t}{A_t} = \frac{S_{at}^\lambda}{A_t^{1-\phi}} \quad (103)$$

Given that S_{at} is a fixed fraction of the total population, the numerator grows at rate $\lambda\bar{n}$. The denominator grows at $(1-\phi)g_{At}$. Given that on a balanced growth path, g_A must be constant, the numerator and denominator must grow at the same rate, yielding

$$g_A = \frac{\lambda\bar{n}}{1-\phi} \quad (104)$$

The same reasoning applies to g_B , giving us

$$g_B = \frac{\lambda\bar{n}}{1-\phi} \quad (105)$$

Now, note that $C_t = A_t^\alpha L_{ct}$. Given that L_{ct} is a fixed fraction of the total population, C_t grows at rate $\alpha g_{At} + \bar{n}$. Given $c_t = C_t/N_t$, c_t grows at rate αg_{At} . Thus, on the balanced growth path,

$$g_c = \alpha g_A = \frac{\alpha\lambda\bar{n}}{1-\phi} \quad (106)$$

The same reasoning applies to g_h , so

$$g_h = \alpha g_B = \frac{\alpha\lambda\bar{n}}{1-\phi} \quad (107)$$

Finally, consider what happens to $\delta_t = \bar{\delta} N_t^{\epsilon-\beta} c_t^\epsilon h_t^{-\beta}$. It follows directly that $g_{\delta t} = (\epsilon - \beta)\bar{n} + \epsilon g_{ct} - \beta g_{ht}$. Thus, on the balanced growth path

$$g_\delta = (\epsilon - \beta)\bar{n} + \epsilon \frac{\alpha\lambda\bar{n}}{1-\phi} - \beta \frac{\alpha\lambda\bar{n}}{1-\phi} \implies g_\delta = (\epsilon - \beta) \left(\frac{\alpha\lambda\bar{n}}{1-\phi} + \bar{n} \right) \quad (108)$$

A.2 First Order Conditions of the Hamiltonian

FOC: s_t

$$\begin{aligned}
0 &= \frac{\partial \mathcal{H}}{\partial s_t} \\
\implies 0 &= \lambda s_t^{\lambda-1} p_{at} \sigma_t^\lambda N_t^\lambda A_t^\phi - \lambda (1-s_t)^{\lambda-1} p_{bt} \sigma_t^\lambda N_t^\lambda B_t^\phi \\
\implies \lambda p_{at} \dot{A}_t s_t^{-1} &= \lambda p_{bt} \dot{B}_t (1-s_t)^{-1} \\
\implies \frac{1-s_t}{s_t} &= \frac{p_{bt} \dot{B}_t}{p_{at} \dot{A}_t} \tag{109}
\end{aligned}$$

FOC: ℓ_t

$$\begin{aligned}
0 &= \frac{\partial \mathcal{H}}{\partial \ell_t} \\
\implies 0 &= \frac{\partial}{\partial \ell_t} (M_t u(c_t) - v_t \delta_t M_t) \\
\implies M_t \frac{\partial}{\partial \ell_t} \left(\bar{u} + \frac{(A_t^\alpha \ell_t (1-\sigma_t))^{1-\gamma}}{1-\gamma} \right) &= M_t v_t \bar{\delta} N_t^{\epsilon-\beta} \frac{\partial}{\partial \ell_t} ([A_t^\alpha \ell_t (1-\sigma_t)]^\epsilon [B_t^\alpha (1-\ell_t)(1-\sigma_t)]^{-\beta}) \\
\implies \frac{(1-\gamma)(A_t^\alpha \ell_t (1-\sigma_t))^{-\gamma} A_t^\alpha (1-\sigma_t)}{1-\gamma} &= v_t \bar{\delta} N_t^{\epsilon-\beta} \epsilon [A_t^\alpha \ell_t (1-\sigma_t)]^{\epsilon-1} A_t^\alpha (1-\sigma_t) \\
&\quad [B_t^\alpha (1-\ell_t)(1-\sigma_t)]^{-\beta} \\
&\quad + v_t \bar{\delta} N_t^{\epsilon-\beta} (-\beta) [B_t^\alpha (1-\ell_t)(1-\sigma_t)]^{-\beta-1} \\
&\quad B_t^\alpha (1-\sigma_t) (-1) [A_t^\alpha \ell_t (1-\sigma_t)]^\epsilon \\
\implies u'(c_t) c_t \ell_t^{-1} &= v_t \bar{\delta} N_t^{\epsilon-\beta} \epsilon \ell_t^{-1} c_t^\epsilon h_t^{-\beta} + v_t \bar{\delta} N_t^{\epsilon-\beta} \beta (1-\ell_t)^{-1} h_t^{-\beta} c_t^\epsilon \\
\implies u'(c_t) c_t \ell_t^{-1} &= v_t \delta_t (\epsilon \ell_t^{-1} + \beta (1-\ell_t)^{-1}) \\
\implies \frac{(1-\ell_t)}{\ell_t} &= \frac{v_t}{u'(c_t) c_t} \delta_t \left(\epsilon \frac{(1-\ell_t)}{\ell_t} + \beta \right) \\
\implies \left(1 - \epsilon \delta_t \frac{v_t}{u'(c_t) c_t} \right) \frac{(1-\ell_t)}{\ell_t} &= \beta \delta_t \frac{v_t}{u'(c_t) c_t} \\
\implies \frac{(1-\ell_t)}{\ell_t} &= \beta \delta_t \frac{v_t}{u'(c_t) c_t} \left(1 - \epsilon \delta_t \frac{v_t}{u'(c_t) c_t} \right)^{-1} \tag{110}
\end{aligned}$$

Consider the term $\frac{v_t}{u'(c_t)c_t}$: it is shadow value of life divided by the value of consumption in util terms. It thus represents the relative value of life, and it is convenient to define this explicitly:

$$\tilde{v}_t \equiv \frac{v_t}{u'(c_t)c_t} \quad (111)$$

giving us:

$$\frac{1 - \ell_t}{\ell_t} = \frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t} \quad (112)$$

Note that this is a very logical condition: the ratio of workers is proportional to what these workers can produce. In the numerator is the hazard rate times the relative value of life times β (the effectiveness of safety goods in reducing existential risk)—this is what can be gained by making a safety good. In the denominator is 1 (which is value of consumption relative to \tilde{v}_t) minus the existential risk increasing effects of consumption.

FOC: σ_t

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}}{\partial \sigma_t} \\ \implies 0 &= M_t \frac{\partial}{\partial \sigma_t} \left(\bar{u} + \frac{(A_t^\alpha \ell_t (1 - \sigma_t))^{1-\gamma}}{1-\gamma} \right) + \lambda \sigma_t^{\lambda-1} p_{at} s_t^\lambda N_t^\lambda A^\phi \\ &\quad + \lambda \sigma_t^{\lambda-1} p_{at} (1 - s_t)^\lambda N_t^\lambda B^\phi - M_t v_t \bar{\delta} N_t^{\epsilon-\beta} \frac{\partial}{\partial \sigma_t} ([A_t^\alpha \ell_t (1 - \sigma_t)]^\epsilon [B_t^\alpha (1 - \ell_t)(1 - \sigma_t)]^{-\beta}) \\ \implies 0 &= M_t \frac{1-\gamma}{1-\gamma} (A_t^\alpha \ell_t (1 - \sigma_t))^{-\gamma} A_t^\alpha \ell_t (-1) + \frac{\lambda p_{at} \dot{A} + \lambda p_{bt} \dot{B}}{\sigma} \\ &\quad - M_t v_t \bar{\delta} N_t^{\epsilon-\beta} \epsilon [A_t^\alpha \ell_t (1 - \sigma_t)]^{\epsilon-1} (-1) A_t^\alpha \ell_t [B_t^\alpha (1 - \ell_t)(1 - \sigma_t)]^{-\beta} \\ &\quad - M_t v_t \bar{\delta} N_t^{\epsilon-\beta} (-\beta) [B_t^\alpha (1 - \ell_t)(1 - \sigma_t)]^{-\beta-1} (-1) B_t^\alpha (1 - \ell_t) [A_t^\alpha \ell_t (1 - \sigma_t)]^\epsilon \\ \implies \frac{M_t u'(c_t) c_t + v_t M_t \beta \delta_t - v_t M_t \epsilon \delta_t}{1 - \sigma_t} &= \frac{\lambda (p_{at} \dot{A} + p_{bt} \dot{B})}{\sigma_t} \\ \implies \frac{\sigma_t}{1 - \sigma_t} &= \frac{\lambda (p_{at} \dot{A} + p_{bt} \dot{B})}{M_t [u'(c_t) c_t + (\beta - \epsilon) \delta_t v_t]} \quad (113) \end{aligned}$$

FOC: M_t

$$\begin{aligned}\rho &= \frac{\partial \mathcal{H} / \partial M_t + \dot{v}_t}{v_t} \\ \implies \rho &= \frac{\dot{v}_t}{v_t} + \frac{1}{v_t} [u(c_t) - v_t \delta_t]\end{aligned}\quad (114)$$

FOC: A_t

$$\begin{aligned}\rho &= \frac{\partial \mathcal{H} / \partial A_t + \dot{p}_{at}}{p_{at}} \\ \implies \rho &= \frac{\dot{p}_{at}}{p_{at}} + \frac{1}{p_{at}} [M_t \frac{1-\gamma}{1-\gamma} (A_t^\alpha \ell_t (1-\sigma_t))^{-\gamma} \alpha A^{\alpha-1} \ell_t (1-\sigma_t) + \phi A^{\phi-1} p_{at} s_t^\lambda \sigma_t^\lambda N_t^\lambda \\ &\quad - M_t v_t \bar{\delta} N_t^{\epsilon-\beta} (B_t^\alpha (1-\ell_t) (1-\sigma_t))^{-\beta} \epsilon (A_t^\alpha \ell_t (1-\sigma_t))^{\epsilon-1} \alpha A^{\alpha-1} \ell_t (1-\sigma_t)] \\ \implies \rho &= \frac{\dot{p}_{at}}{p_{at}} + \frac{1}{p_{at}} [M_t u'(c_t) \alpha \frac{c_t}{A_t} + p_{at} \phi \frac{\dot{A}_t}{A_t} - \alpha \epsilon v_t M_t \frac{\delta_t}{A_t}]\end{aligned}\quad (115)$$

FOC: B_t

$$\begin{aligned}\rho &= \frac{\partial \mathcal{H} / \partial B_t + \dot{p}_{bt}}{p_{bt}} \\ \implies \rho &= \frac{\dot{p}_{bt}}{p_{bt}} + \frac{1}{p_{bt}} [\phi B^{\phi-1} p_{bt} (1-s_t)^\lambda \sigma_t^\lambda N_t^\lambda \\ &\quad - M_t v_t \bar{\delta} N_t^{\epsilon-\beta} (A_t^\alpha \ell_t (1-\sigma_t))^\epsilon (B_t^\alpha (1-\ell_t) (1-\sigma_t))^{-\beta-1} (-\beta) \alpha B^{\alpha-1} (1-\ell_t) (1-\sigma_t)] \\ \implies \rho &= \frac{\dot{p}_{bt}}{p_{bt}} + \frac{1}{p_{bt}} [p_{bt} \phi \frac{\dot{B}_t}{B_t} + \alpha \beta v_t M_t \frac{\delta_t}{B_t}]\end{aligned}\quad (116)$$

Transversality Conditions

Note that the three standard transversality conditions apply:

$$\lim_{t \rightarrow \infty} [e^{-\rho t} \cdot v_t M_t] = 0 \quad (117)$$

$$\lim_{t \rightarrow \infty} [e^{-\rho t} \cdot p_{at} A_t] = 0 \quad (118)$$

$$\lim_{t \rightarrow \infty} [e^{-\rho t} \cdot p_{bt} B_t] = 0 \quad (119)$$

The Price of Ideas

To solve for the allocation of scientists (see FOC: s_t), I need to solve for the relative price of ideas p_{bt}/p_{at} . To do this, I manipulate FOC B_t :

$$\begin{aligned}\rho &= \frac{\dot{p}_{bt}}{p_{bt}} + \frac{1}{p_{bt}} \left[p_{bt} \phi \frac{\dot{B}_t}{B_t} + \alpha \beta v_t M_t \frac{\delta_t}{B_t} \right] \\ \implies \rho - \frac{\dot{p}_{bt}}{p_{bt}} - \phi \frac{\dot{B}_t}{B_t} &= \frac{1}{p_{bt}} \left[\alpha \beta v_t M_t \frac{\delta_t}{B_t} \right] \\ \implies p_{bt} &= \frac{\alpha \beta v_t M_t \delta_t / B_t}{\rho - g_{p_{bt}} - \phi g_{B_t}}\end{aligned}\tag{120}$$

Similarly, I manipulate FOC A_t :

$$\begin{aligned}\rho &= \frac{\dot{p}_{at}}{p_{at}} + \frac{1}{p_{at}} \left[M_t u'(c_t) \alpha \frac{c_t}{A_t} + p_{at} \phi \frac{\dot{A}_t}{A_t} - \alpha \epsilon v_t M_t \frac{\delta_t}{A_t} \right] \\ \implies \rho - \frac{\dot{p}_{at}}{p_{at}} - \phi \frac{\dot{A}_t}{A_t} &= \frac{1}{p_{at}} \left[M_t u'(c_t) \alpha \frac{c_t}{A_t} - \alpha \epsilon v_t M_t \frac{\delta_t}{A_t} \right] \\ \implies p_{at} &= \frac{\alpha M_t (u'(c_t) c_t - \epsilon \delta_t v_t) / A_t}{\rho - g_{p_{at}} - \phi g_{A_t}}\end{aligned}\tag{121}$$

Combining the two, the relative price must satisfy:

$$\frac{p_{bt} B_t}{p_{at} A_t} = \frac{\beta \delta_t v_t}{u'(c_t) c_t - \epsilon \delta_t v_t} \cdot \frac{\rho - g_{p_{at}} - \phi g_{A_t}}{\rho - g_{p_{bt}} - \phi g_{B_t}}\tag{122}$$

Putting this in terms of the previously defined relative value of life \tilde{v}_t I get:

$$\frac{p_{bt} B_t}{p_{at} A_t} = \frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t} \cdot \frac{\rho - g_{p_{at}} - \phi g_{A_t}}{\rho - g_{p_{bt}} - \phi g_{B_t}}\tag{123}$$

There needs to be a condition on ρ to keep the denominators positive.

Allocation of Scientists

Recall from FOC: s_t that:

$$\frac{1 - s_t}{s_t} = \frac{p_{bt}\dot{B}}{p_{at}\dot{A}}$$

I can now substitute in the relative price of ideas:

$$\frac{1 - s_t}{s_t} = \frac{\beta\delta_t\tilde{v}_t}{1 - \epsilon\delta_t\tilde{v}_t} \cdot \frac{\rho - g_{p_{at}} - \phi g_{At}}{\rho - g_{p_{bt}} - \phi g_{Bt}} \cdot \frac{g_{Bt}}{g_{At}} \quad (124)$$

Recall from FOC: ℓ_t that $(1 - \ell_t)/\ell_t = (\beta\delta_t\tilde{v}_t)/(1 - \epsilon\delta_t\tilde{v}_t)$, so both of these key allocation variables depend on $\delta_t\tilde{v}_t$, that is, on the race between the decline in the hazard rate and the possible rise in the value of life relative to consumption.

Note that in Jones (2016), $(1 - \ell_t)/\ell_t$ and $(1 - s_t)/s_t$ are instead proportional simply to $\beta\delta_t\tilde{v}_t$. Incorporating the existential risk effects of higher consumption, there is a (much) higher allocation of labor and of scientists to safety, in particular in the case that the value of life rises faster than the hazard rate falls, i.e. $\delta_t\tilde{v}_t$ rises.

Moreover, our model introduces an additional constraint. Since ℓ_t is the fraction of labor allocated to consumption, it must be that $0 < \ell_t \leq 1$ (where the strict inequality comes from the fact that at least some labor must be allocated to consumption along the balanced growth path). Thus, $\frac{(1-\ell_t)}{\ell_t}$ must be finite, i.e. the denominator cannot be 0. Given that ϵ , β , δ_t , and \tilde{v}_t are guaranteed to be positive, along the optimal path:

$$\delta_t\tilde{v}_t < \frac{1}{\epsilon} \quad (125)$$

This foreshadows what will happen along the balanced growth path: given the parameters of our preferences, either δ_t falls to 0 more quickly than \tilde{v}_t grows, in which case $\delta_t\tilde{v}_t$ falls to 0, or $\delta_t\tilde{v}_t$ asymptotically approaches $1/\epsilon$.

Characterizing \tilde{v}_t

Using FOC: M_t , I obtain

$$\begin{aligned} \rho &= \frac{\dot{v}_t}{v_t} + \frac{1}{v_t}[u(c_t) - v_t\delta_t] \\ \implies \rho - \frac{\dot{v}_t}{v_t} + \delta_t &= \frac{u(c_t)}{v_t} \\ \implies v_t &= \frac{u(c_t)}{\rho + \delta_t - g_{vt}} \end{aligned} \tag{126}$$

$$\implies \tilde{v}_t = \frac{u(c_t)/u'(c_t)c_t}{\rho + \delta_t - g_{vt}} \tag{127}$$

Thus, the relative value of life depends on the extra utility a person enjoys versus increasing consumption on the current margin—this is why the degree of diminishing returns, γ , in our utility function plays such a key role.

Given our isoelastic CRRA utility,

$$\begin{aligned} \frac{u(c_t)}{u'(c_t)c_t} &= \frac{\bar{u} + \frac{c_t^{1-\gamma}}{1-\gamma}}{c_t^{-\gamma}c_t} \\ \implies \frac{u(c_t)}{u'(c_t)c_t} &= \left(\bar{u} + \frac{c_t^{1-\gamma}}{1-\gamma}\right)(c_t^{-(1-\gamma)}) \\ \implies \frac{u(c_t)}{u'(c_t)c_t} &= \bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma} \end{aligned} \tag{128}$$

A.3 Proof of Proposition 2

First, given equations (128) and (127) and $\gamma > 1$, along a balanced growth path in which $c_t \rightarrow \infty$:

$$\begin{aligned} g_{\tilde{v}} &= g_{\frac{u(c_t)}{u'(c_t)c_t}} - g_{\rho+\delta_t-g_{vt}} \\ g_{\tilde{v}} &= g_{\bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma}} \\ g_{\tilde{v}} &= (\gamma - 1)g_c \end{aligned} \tag{129}$$

as long as δ_t converges to some constant.

I shall now conjecture that the solution for the balanced growth path takes the following form: s_t and ℓ_t fall toward zero at a constant exponential rate, while $\sigma_t \rightarrow \sigma^*$. The key condition for this result will be $\gamma > 1 + (\beta - \epsilon) \left(\frac{1-\phi}{\alpha\lambda} + 1 \right)$.

Given $c_t = A_t^\alpha \ell_t (1 - \sigma_t)$, taking logs and derivatives, in our proposed solution consumption growth is given by:

$$g_c = \alpha g_A + g_\ell \tag{130}$$

Now, observe in (124) that s_t is inversely proportional to $\frac{\beta\delta_t\tilde{v}_t}{1-\epsilon\delta_t\tilde{v}_t}$, and that the remaining terms in (124) will be constant along a balanced growth path. Observe in (FOC: ℓ_t) that ℓ_t is also inversely proportional to $\frac{\beta\delta_t\tilde{v}_t}{1-\epsilon\delta_t\tilde{v}_t}$. Thus, along the balanced growth path, $g_\ell = g_s$ and I get:

$$g_c = \alpha g_A + g_s \tag{131}$$

The growth rates of A and B follow straightforwardly from their production functions. Given $\dot{A}_t = s_t^\lambda \sigma_t^\lambda N_t^\lambda A_t^\phi$, $g_{At} = \frac{\dot{A}_t}{A_t} = \frac{s_t^\lambda \sigma_t^\lambda N_t^\lambda}{A_t^{1-\phi}}$, which becomes constant along a balanced growth path, so the numerator and denominator must grow at the same rate:

$$\begin{aligned} \lim_{t \rightarrow \infty} \ln(s_t^\lambda \sigma_t^\lambda N_t^\lambda) &= \lim_{t \rightarrow \infty} \ln(A_t^{1-\phi}) \\ \implies \lambda(g_s + \bar{n}) &= (1 - \phi)g_A \\ \implies g_A &= \frac{\lambda(g_s + \bar{n})}{1 - \phi} \end{aligned} \tag{132}$$

Given $\dot{B}_t = (1 - s_t)^\lambda \sigma_t^\lambda N_t^\lambda B_t^\phi$, $g_{Bt} = \frac{\dot{B}_t}{B_t} = \frac{(1-s_t)^\lambda \sigma_t^\lambda N_t^\lambda}{B_t^{1-\phi}}$, which becomes constant a balanced growth path, so the numerator and denominator must

grow at the same rate. The key difference to A here is that s_t falling to 0 at a constant exponential rate means that $1 - s_t$ will converge to 1 and be asymptotically constant, i.e. $\lim_{t \rightarrow \infty} g_{1-s_t} = 0$.

$$\begin{aligned} \lim_{t \rightarrow \infty} \dot{\ln}((1 - s_t)^\lambda \sigma_t^\lambda N_t^\lambda) &= \lim_{t \rightarrow \infty} \dot{\ln}(B_t^{1-\phi}) \\ \implies \lambda \bar{n} &= (1 - \phi) g_B \\ \implies g_B &= \frac{\lambda \bar{n}}{1 - \phi} \end{aligned} \tag{133}$$

Plugging (132) into (131) I thus get:

$$g_c = \alpha \frac{\lambda(g_s + \bar{n})}{1 - \phi} + g_s \tag{134}$$

Plugging this into (129):

$$g_{\tilde{v}} = (\gamma - 1) \left[\frac{\alpha \lambda (g_s + \bar{n})}{1 - \phi} + g_s \right] \tag{135}$$

Now make a key observation. Recall from FOC: ℓ_t that $(1 - \ell_t)/\ell_t = (\beta \delta_t \tilde{v}_t)/(1 - \epsilon \delta_t \tilde{v}_t)$ (and the allocation of scientists is proportional to this as well). Given a constant, positive ϵ and β , the only way for ℓ_t (and s_t) to fall to 0 is for $\delta_t \tilde{v}_t$ to grow. However, remember (125): $\delta_t \tilde{v}_t < 1/\epsilon$. Thus, as $t \rightarrow \infty$, $\delta_t \tilde{v}_t \rightarrow 1/\epsilon$, i.e. $\delta_t \tilde{v}_t$ is asymptotically constant. However, this in turn means that $\epsilon \delta_t \tilde{v}_t$ converges to 1 asymptotically, meaning that $1 - \epsilon \delta_t \tilde{v}_t$ will fall to 0 exponentially. This then delivers the desired exponential increase in $(1 - \ell_t)/\ell_t$ and the exponential fall to 0 of ℓ_t (and s_t).

This convergence of $\delta_t \tilde{v}_t \rightarrow 1/\epsilon$ is unique to this model. In Jones (2016), given sufficient curvature of preferences, $\delta_t \tilde{v}_t$ goes to infinity. However, this convergence is very logical: in the denominator of our condition for $(1 - \ell_t)/\ell_t$ is the marginal product of consumption labor, $1 - \epsilon \delta_t \tilde{v}_t$. 1 is the normalized value of consumption, whereas $-\epsilon \delta_t \tilde{v}_t$ is the relative impact of consumption on life. Were $\delta_t \tilde{v}_t$ to keep rising above $1/\epsilon$, the marginal product of consumption labor would be negative: consumption labor would destroy life more than it increases utility.

Thus, I know that:

$$\begin{aligned} \lim_{t \rightarrow \infty} \dot{\ln}(\delta_t \tilde{v}_t) &= 0 \\ \implies g_\delta &= -g_{\tilde{v}} \end{aligned} \quad (136)$$

Plugging in (135):

$$g_\delta = -(\gamma - 1) \left[\frac{\alpha \lambda (g_s + \bar{n})}{1 - \phi} + g_s \right] \quad (137)$$

I thus need an expression for g_δ . Given $\delta_t = \bar{\delta} N_t^{\epsilon - \beta} [A_t^\alpha \ell_t (1 - \sigma_t)]^\epsilon [B_t^\alpha (1 - \ell_t)(1 - \sigma_t)]^{-\beta}$:

$$\begin{aligned} g_\delta &= \lim_{t \rightarrow \infty} \dot{\ln}(N_t^{\epsilon - \beta} [A_t^\alpha \ell_t (1 - \sigma_t)]^\epsilon [B_t^\alpha (1 - \ell_t)(1 - \sigma_t)]^{-\beta}) \\ \implies g_\delta &= \lim_{t \rightarrow \infty} \left[(\epsilon - \beta) \dot{\ln}(N_t) + \epsilon \dot{\ln}([A_t^\alpha \ell_t (1 - \sigma_t)]) - \beta \dot{\ln}([B_t^\alpha (1 - \ell_t)(1 - \sigma_t)]) \right] \\ \implies g_\delta &= (\epsilon - \beta) \bar{n} + \epsilon(\alpha g_A + g_\ell) - \beta \alpha g_B \\ \implies g_\delta &= \alpha(\epsilon g_A - \beta g_B) + \epsilon g_\ell + (\epsilon - \beta) \bar{n} \\ \implies g_\delta &= \alpha(\epsilon g_A - \beta g_B) + \epsilon g_s + (\epsilon - \beta) \bar{n} \end{aligned} \quad (138)$$

where I substitute in $g_s = g_\ell$ as explained earlier.

I plug this in and solve:

$$\begin{aligned} \alpha \left(\epsilon \frac{\lambda (g_s + \bar{n})}{1 - \phi} - \beta \frac{\lambda \bar{n}}{1 - \phi} \right) + \epsilon g_s + (\epsilon - \beta) \bar{n} &= -(\gamma - 1) \left[\frac{\alpha \lambda (g_s + \bar{n})}{1 - \phi} + g_s \right] \\ \implies g_s \left(\epsilon + \gamma - 1 + \frac{\alpha \epsilon \lambda + \alpha \lambda (\gamma - 1)}{1 - \phi} \right) &= \frac{-\alpha \lambda \epsilon \bar{n} + \alpha \beta \lambda \bar{n} - (\gamma - 1) \alpha \lambda \bar{n} - (1 - \phi)(\epsilon - \beta) \bar{n}}{1 - \phi} \\ \implies g_s &= \frac{-\alpha \lambda \epsilon \bar{n} + \alpha \beta \lambda \bar{n} - (\gamma - 1) \alpha \lambda \bar{n} - (1 - \phi)(\epsilon - \beta) \bar{n}}{(1 - \phi)(\epsilon + \gamma - 1) + \alpha \epsilon \lambda + \alpha \lambda (\gamma - 1)} \\ \implies g_s &= \frac{\alpha \lambda \bar{n} (-\epsilon + \beta - \gamma + 1) + (1 - \phi)(\epsilon - \beta) \bar{n}}{(1 - \phi)(\epsilon + \gamma - 1) + \alpha \lambda (\epsilon + \gamma - 1)} \\ \implies g_s &= \frac{\bar{n} [\alpha \lambda (1 + \beta - \epsilon - \gamma) + (1 - \phi)(\beta - \epsilon)]}{(\gamma + \epsilon - 1)(\alpha \lambda - \phi + 1)} \end{aligned} \quad (139)$$

g_s is negative

$$\begin{aligned}
&\iff \alpha\lambda(1 + \beta - \epsilon - \gamma) + (1 - \phi)(\beta - \epsilon) < 0 \\
&\iff \alpha\lambda(\gamma - 1 - \beta + \epsilon) > (1 - \phi)(\beta - \epsilon) \\
&\iff \frac{\alpha\lambda}{1 - \phi}((\gamma - 1) - (\beta - \epsilon)) > \beta - \epsilon \\
&\iff \frac{\alpha\lambda}{1 - \phi}(\gamma - 1) > (\beta - \epsilon) \left(1 + \frac{\alpha\lambda}{1 - \phi}\right) \\
&\iff \gamma > 1 + (\beta - \epsilon) \left(\frac{1 - \phi}{\alpha\lambda} + 1\right)
\end{aligned} \tag{140}$$

Thus, $\gamma > 1 + (\beta - \epsilon) \left(\frac{1 - \phi}{\alpha\lambda} + 1\right)$ is the key condition delivering this balanced growth path.

I can now calculate the other asymptotic growth rates as well. It will be helpful to define:

$$\bar{g} \equiv \frac{\alpha\lambda\bar{n}}{1 - \phi} \tag{141}$$

The asymptotic convergence of $1 - \ell_t$ directly implies:

$$g_h = \alpha g_B = \frac{\alpha\lambda\bar{n}}{1 - \phi} = \bar{g} \tag{142}$$

I can put g_s in terms of \bar{g} :

$$\begin{aligned}
g_s &= \frac{\alpha\lambda\bar{n}(1 + \beta - \epsilon - \gamma) + \bar{n}(1 - \phi)(\beta - \epsilon)}{(\gamma + \epsilon - 1)(1 - \phi) + (\gamma + \epsilon - 1)(\alpha\lambda)} \\
\implies g_s &= -\bar{g} \cdot \frac{\gamma - 1 - \beta + \epsilon}{\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)(\gamma + \epsilon - 1)} - \bar{n} \cdot \frac{\epsilon - \beta}{\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)(\gamma + \epsilon - 1)}
\end{aligned} \tag{143}$$

I can now calculate g_c from (134):

$$\begin{aligned}
g_c &= \alpha \frac{\lambda(g_s + \bar{n})}{1 - \phi} + g_s \\
\Rightarrow g_c &= \frac{\alpha \lambda \bar{n}}{1 - \phi} + \left(1 + \frac{\alpha \lambda}{1 - \phi}\right) \cdot g_s \\
\Rightarrow g_c &= \bar{g} \cdot \left[1 - \frac{(\gamma - (1 + \beta - \epsilon)) + \frac{1-\phi}{\alpha \lambda}(\epsilon - \beta) + \frac{\alpha \lambda}{1-\phi}(\gamma - (1 + \beta - \epsilon)) + (\epsilon - \beta)}{\left(1 + \frac{\alpha \lambda}{1-\phi}\right)(\gamma + \epsilon - 1)} \right] \\
\Rightarrow g_c &= \bar{g} \cdot \left[1 - \frac{\left(1 + \frac{\alpha \lambda}{1-\phi}\right)(\gamma - (1 + \beta - \epsilon)) + \left(1 + \frac{1-\phi}{\alpha \lambda}\right)(\epsilon - \beta)}{\left(1 + \frac{\alpha \lambda}{1-\phi}\right)(\gamma + \epsilon - 1)} \right] \\
\Rightarrow g_c &= \bar{g} \cdot \left[1 - \frac{(\gamma - (1 + \beta - \epsilon)) + (\epsilon - \beta) \frac{1 + \frac{1-\phi}{\alpha \lambda}}{1 + \frac{1-\phi}{\alpha \lambda}}}{\gamma - 1 + \epsilon} \right] \\
\Rightarrow g_c &= \bar{g} \cdot \left[1 - \frac{(\gamma - (1 + \beta - \epsilon)) + (\epsilon - \beta) \frac{1-\phi}{\alpha \lambda}}{\gamma - 1 + \epsilon} \right] \\
\Rightarrow g_c &= \bar{g} \cdot \left[\frac{\beta + (\beta - \epsilon) \frac{1-\phi}{\alpha \lambda}}{\gamma + \epsilon - 1} \right] \tag{144}
\end{aligned}$$

Since $g_s < 0$, $g_c < \bar{g}$. $g_c > 0$ follows directly when $\beta \geq \epsilon$, as in this case.

How Low Can ρ Go?

What values of ρ are permissible for our asymptotic growth path to be valid?

In particular, the denominators of our shadow prices must be positive and the optimal allocations must satisfy the transversality conditions.

First, consider p_{bt} . Recall that

$$p_{bt} = \frac{\alpha \beta v_t M_t \delta_t / B_t}{\rho - g_{p_b t} - \phi g_{Bt}}$$

The denominator has to be positive along the balanced growth path and $g_{p_b t} \rightarrow g_{p_b}$. Then, if the denominator is positive and thus asymptotically constant along the balanced growth path:

$$g_{p_b} = \lim_{t \rightarrow \infty} \dot{\ln}(M_t) + \dot{\ln}(v_t \delta_t) - g_B$$

Since δ_t converges to 0, $g_M = 0$. Moreover, recall that $\delta_t v_t = \delta_t \tilde{v}_t u'(c_t) c_t$, so $g_{\delta v} = g_{\delta \tilde{v}} + g_{u'(c)c} = g_{u'(c)c} = (1 - \gamma)g_c$ since $\delta_t \tilde{v}_t$ is asymptotically constant, so

$$g_{p_b} = (1 - \gamma)g_c - g_B \quad (145)$$

The condition that the denominator of g_{p_b} is positive and asymptotically constant along the balanced growth path now becomes:

$$\rho > g_{p_b} + \phi g_B = (1 - \gamma)g_c + (\phi - 1)g_B \quad (146)$$

Given $\gamma > 1$ and $\phi < 1$ the right hand side is negative, meaning any $\rho \geq 0$ is valid.

Recall the transversality condition for B_t :

$$\lim_{t \rightarrow \infty} [e^{-\rho t} \cdot p_{bt} \cdot B_t] = 0$$

Note that since $\gamma > 1$, $-g_{p_b} > g_B$, so the transversality condition is satisfied even for $\rho = 0$

Now, consider p_{at} . Recall that

$$p_{at} = \frac{\alpha M_t(u'(c_t)c_t - \epsilon \delta_t v_t) / A_t}{\rho - g_{p_{at}} - \phi g_{A_t}}$$

The denominator has to be positive along the balanced growth path and $g_{p_{at}} \rightarrow g_{p_a}$. Then, if the denominator is positive and thus asymptotically constant along the balanced growth path:

$$g_{p_a} = \lim_{t \rightarrow \infty} \dot{\ln}(u'(c_t)c_t - \epsilon \delta_t v_t) - g_A$$

given $g_M = 0$. Again, $\delta_t v_t = \delta_t \tilde{v}_t u'(c_t) c_t$, so

$$\begin{aligned} g_{p_a} &= \lim_{t \rightarrow \infty} \dot{\ln}(1 - \epsilon \delta_t \tilde{v}_t) + \dot{\ln}(u'(c_t)c_t) - g_A \\ \implies g_{p_a} &= \lim_{t \rightarrow \infty} \dot{\ln}(1 - \epsilon \delta_t \tilde{v}_t) + (1 - \gamma)g_c - g_A \end{aligned}$$

As $\epsilon\delta_t\tilde{v}_t$ converges to $1/\epsilon$ along the balanced growth path, $(1 - \epsilon\delta_t\tilde{v}_t)$ falls exponentially to zero. Indeed, note that since $1 - s_t$ and $\beta\delta_t\tilde{v}_t$ are asymptotically constant, $g_s = -g_{\beta\delta\tilde{v}/(1-\epsilon\delta\tilde{v})} = g_{1-\epsilon\delta\tilde{v}}$, so

$$g_{p_a} = g_s + (1 - \gamma)g_c - g_A \quad (147)$$

The condition that the denominator of g_{p_a} be positive and asymptotically constant now becomes

$$\rho > g_{p_a} + \phi g_A = g_s + (1 - \gamma)g_c + (\phi - 1)g_A \quad (148)$$

Again, since $g_s < 0$, $\gamma > 1$, $\phi < 1$, $g_c > 0$, and $g_A > 0$, the right hand side is negative and any $\rho \geq 0$ is valid.

Recall the transversality condition for A_t :

$$\lim_{t \rightarrow \infty} [e^{-\rho t} \cdot p_{at} \cdot A_t] = 0$$

Since $g_s < 0$ and $\gamma > 1$, I get $-g_{p_a} > g_A$, satisfying the transversality condition for A even for $\rho = 0$.

Next, recall the final transversality condition

$$\lim_{t \rightarrow \infty} [e^{-\rho t} \cdot v_t \cdot M_t] = 0$$

Since $g_M = 0$, $M_t \rightarrow M_t^* > 0$. Thus, either g_v falls exponentially to zero or $\rho > 0$ for the transversality condition to hold.

Recall (126):

$$v_t = \frac{u(c_t)}{\rho + \delta_t - g_{vt}}$$

$u(c_t) \rightarrow \bar{u}$ and $g_{vt} \rightarrow g_v$ along a balanced growth path. Thus, given a positive denominator, the denominator is asymptotically constant as $\delta_t \rightarrow 0$, implying

$$g_v = 0 \quad (149)$$

Since δ_t falls exponentially to zero and $g_v = 0$, the denominator is positive if and only if $\rho > 0$. $\rho > 0$ then also ensures the transversality condition holds.

Thus, our balanced growth path is a valid solution for any $\rho > 0$.

Note that this ρ is still not considering population growth, i.e. considers a somewhat selfish agent.

Finally, I have to find σ^* . Substituting the prices of ideas into (FOC: σ_t) I get:

$$\begin{aligned} \frac{\sigma_t}{1 - \sigma_t} &= \frac{\lambda(p_{at}\dot{A} + p_{bt}\dot{B})}{M_t[u'(c_t)c_t + (\beta - \epsilon)\delta_t v_t]} \\ \implies \frac{\sigma_t}{1 - \sigma_t} &= \frac{\lambda\left(\frac{\alpha M_t(u'(c_t)c_t - \epsilon\delta_t v_t)/A_t}{\rho - g_{pat} - \phi g_{At}}\dot{A} + \frac{\alpha\beta v_t M_t \delta_t / B_t}{\rho - g_{pbt} - \phi g_{Bt}}\dot{B}\right)}{M_t[u'(c_t)c_t + (\beta - \epsilon)\delta_t v_t]} \\ \implies \frac{\sigma_t}{1 - \sigma_t} &= \lambda\alpha(g_{At}\frac{\frac{u'(c_t)c_t - \epsilon\delta_t v_t}{\rho - g_{pat} - \phi g_{At}}}{u'(c_t)c_t + (\beta - \epsilon)\delta_t v_t} + g_{Bt}\frac{\frac{\beta v_t \delta_t}{\rho - g_{pbt} - \phi g_{Bt}}}{u'(c_t)c_t + (\beta - \epsilon)\delta_t v_t}) \end{aligned}$$

Now, recall that $\delta_t \tilde{v}_t \rightarrow 1/\epsilon$, so $\delta_t v_t = \delta_t \tilde{v}_t \cdot u'(c_t)c_t \rightarrow 1/\epsilon \cdot u'(c_t)c_t$.

$$\begin{aligned} \implies \frac{\sigma^*}{1 - \sigma^*} &= \lim_{t \rightarrow \infty} \lambda\alpha(g_{At}\frac{\frac{u'(c_t)c_t - u'(c_t)c_t}{\rho - g_{pat} - \phi g_{At}}}{(\beta/\epsilon)u'(c_t)c_t} + g_{Bt}\frac{\frac{(\beta/\epsilon)u'(c_t)c_t}{\rho - g_{pbt} - \phi g_{Bt}}}{(\beta/\epsilon)u'(c_t)c_t}) \\ \implies \frac{\sigma^*}{1 - \sigma^*} &= \lim_{t \rightarrow \infty} \frac{\lambda\alpha g_{Bt}}{\rho - g_{pbt} - \phi g_{Bt}} \end{aligned} \quad (150)$$

I can now substitute in g_{p_b} along our balanced growth path.

$$\frac{\sigma^*}{1 - \sigma^*} = \frac{\lambda\alpha g_B}{\rho + (\gamma - 1)g_c + (1 - \phi)g_B} \quad (151)$$

Therefore,

$$\begin{aligned} \sigma^* \left[1 + \frac{\lambda\alpha g_B}{\rho + (\gamma - 1)g_c + (1 - \phi)g_B} \right] &= \frac{\lambda\alpha g_B}{\rho + (\gamma - 1)g_c + (1 - \phi)g_B} \\ \implies \sigma^* &= \frac{\frac{\lambda\alpha g_B}{\rho + (\gamma - 1)g_c + (1 - \phi)g_B}}{1 + \frac{\lambda\alpha g_B}{\rho + (\gamma - 1)g_c + (1 - \phi)g_B}} \\ \implies \sigma^* &= \frac{\lambda\alpha g_B}{\rho + (\gamma - 1)g_c + (1 - \phi)g_B + \lambda\alpha g_B} \\ \implies \sigma^* &= \frac{\lambda\alpha g_B}{\rho + (\gamma - 1)g_c + (1 - \phi + \lambda\alpha)g_B} \end{aligned}$$

A.4 Proof of Proposition 3

I conjecture that $\tilde{s}_t \equiv 1 - s_t$ and $\tilde{\ell}_t \equiv 1 - \ell_t$ fall exponentially to zero on the asymptotic growth path, while $\sigma_t \rightarrow \sigma^*$. Then, it follows directly that

$$g_h = \alpha g_B + g_{\tilde{\ell}} \quad (152)$$

Moreover, since $(1 - \ell_t)/\ell_t = \tilde{\ell}_t/(1 - \tilde{\ell}_t)$ and $(1 - s_t)/s_t = \tilde{s}_t/(1 - \tilde{s}_t)$ are both proportional to $(\beta\delta_t\tilde{v}_t)/(1 - \epsilon\delta_t\tilde{v}_t)$ along an asymptotic growth path, implying $g_{\tilde{s}} = g_{\tilde{\ell}}$ along the asymptotic growth path (analogous to $g_s = g_\ell$ along the asymptotic growth path in the proof of proposition 2). Thus,

$$g_h = \alpha g_B + g_{\tilde{s}} \quad (153)$$

Moreover, an asymptotically constant g_B requires

$$g_B = \frac{\lambda(\bar{n} + g_{\tilde{s}})}{1 - \phi} \quad (154)$$

thus implying

$$g_h = \alpha \frac{\lambda(\bar{n} + g_{\tilde{s}})}{1 - \phi} + g_{\tilde{s}} \quad (155)$$

Since $\tilde{\ell}_t \rightarrow 0$, ℓ_t is asymptotically constant,

$$g_c = \alpha g_A \quad (156)$$

Similarly, s_t is asymptotically constant, so an asymptotically constant g_A then directly requires that

$$g_A = \frac{\lambda\bar{n}}{1 - \phi} \quad (157)$$

thus implying

$$g_c = \alpha \frac{\lambda\bar{n}}{1 - \phi} \quad (158)$$

Now, notice that for \tilde{s}_t to fall to zero exponentially, $(1 - s_t)/s_t = \tilde{s}_t/(1 - \tilde{s}_t)$ has to fall exponentially to zero. On the asymptotic growth path $\tilde{s}_t/(1 - \tilde{s}_t)$ is

proportional to $(\beta\delta_t\tilde{v}_t)/(1-\epsilon\delta_t\tilde{v}_t)$, so for \tilde{s}_t to fall to zero exponentially, $\delta_t\tilde{v}_t$ has to fall to zero exponentially, meaning $1-\epsilon\delta_t\tilde{v}_t$ is asymptotically constant. Thus, $g_{(\beta\delta_t\tilde{v}_t)/(1-\epsilon\delta_t\tilde{v}_t)} = g_{\delta_t\tilde{v}_t} = g_{\tilde{s}}$. Thus,

$$g_{\tilde{s}} = g_{\delta} + g_{\tilde{v}} \quad (159)$$

It follows that

$$\begin{aligned} g_{\delta} &= \epsilon g_c - \beta g_h + (\epsilon - \beta)\bar{n} \\ \implies g_{\delta} &= \epsilon\alpha\frac{\lambda\bar{n}}{1-\phi} - \beta\alpha\frac{\lambda(\bar{n} + g_{\tilde{s}})}{1-\phi} - \beta g_{\tilde{s}} + (\epsilon - \beta)\bar{n} \end{aligned} \quad (160)$$

To get an expression for $g_{\tilde{v}}$, I must differentiate between the cases in which $\gamma \leq 1$ and $\gamma > 1$. Note that

$$\begin{aligned} g_{\tilde{v}} &= g_{\frac{u(c_t)}{u'(c_t)c_t}} - g_{\rho+\delta_t-g_{vt}} \\ g_{\tilde{v}} &= g_{\bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma}} \end{aligned}$$

as long as δ_t converges to a constant

Thus, when $\gamma > 1$, $g_{\tilde{v}} = (\gamma - 1)g_c$, while when $\gamma \leq 1$, \tilde{v} is asymptotically constant so $g_{\tilde{v}} = 0$. I will consider the $\gamma > 1$ case first. Then,

$$\begin{aligned} g_{\tilde{s}} &= g_{\delta} + g_{\tilde{v}} \\ \implies g_{\tilde{s}} &= \epsilon\alpha\frac{\lambda\bar{n}}{1-\phi} - \beta\alpha\frac{\lambda(\bar{n} + g_{\tilde{s}})}{1-\phi} - \beta g_{\tilde{s}} + (\epsilon - \beta)\bar{n} + (\gamma - 1)\alpha\frac{\lambda\bar{n}}{1-\phi} \\ \implies g_{\tilde{s}} &= \frac{\alpha\lambda\bar{n}}{1-\phi}(\epsilon - \beta + \gamma - 1) - \beta g_{\tilde{s}}\left(1 + \frac{\alpha\lambda}{1-\phi}\right) + (\epsilon - \beta)\bar{n} \\ \implies g_{\tilde{s}}\left(1 + \beta + \beta\frac{\alpha\lambda}{1-\phi}\right) &= \frac{\alpha\lambda\bar{n}}{1-\phi}(\epsilon - \beta + \gamma - 1) + (\epsilon - \beta)\bar{n} \\ \implies g_{\tilde{s}} &= \frac{\bar{n}[\alpha\lambda(\epsilon - \beta + \gamma - 1) + (1-\phi)(\epsilon - \beta)]}{(1+\beta)(1-\phi) + \beta\alpha\lambda} \\ \implies g_{\tilde{s}} &= \frac{-\bar{n}\left[\frac{\alpha\lambda}{1-\phi}(1-\epsilon + \beta - \gamma) + (\beta - \epsilon)\right]}{1 + \beta\left(1 + \frac{\alpha\lambda}{1-\phi}\right)} \end{aligned} \quad (161)$$

Then, the condition for $g_{\bar{s}}$ to be negative is

$$\begin{aligned}
& \alpha\lambda(\epsilon - \beta + \gamma - 1) + (1 - \phi)(\epsilon - \beta) < 0 \\
& \iff \frac{\alpha\lambda}{1 - \phi}((\gamma - 1) - (\beta - \epsilon)) < (\beta - \epsilon) \\
& \iff \frac{\alpha\lambda}{1 - \phi}(\gamma - 1) < (\beta - \epsilon) \left(1 + \frac{\alpha\lambda}{1 - \phi}\right) \\
& \iff \gamma < 1 + (\beta - \epsilon) \left(\frac{1 - \phi}{\alpha\lambda} + 1\right)
\end{aligned} \tag{162}$$

which is the key condition delivering our result.

Let $\bar{g} \equiv \frac{\alpha\lambda\bar{n}}{1 - \phi}$. I can now calculate g_h :

$$\begin{aligned}
g_h &= \frac{\alpha\lambda\bar{n}}{1 - \phi} + \left(1 + \frac{\alpha\lambda}{1 - \phi}\right) g_{\bar{s}} \\
\implies g_h &= \frac{\alpha\lambda\bar{n}}{1 - \phi} + \left(1 + \frac{\alpha\lambda}{1 - \phi}\right) \frac{\bar{n} \left[\frac{\alpha\lambda}{1 - \phi}(\epsilon - \beta + \gamma - 1) + (\epsilon - \beta) \right]}{1 + \beta\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)} \\
\implies g_h &= \bar{g} \left(1 + \frac{\frac{\alpha\lambda}{1 - \phi}(\epsilon - \beta + \gamma - 1) + (\epsilon - \beta) + (\epsilon - \beta + \gamma - 1) + \frac{1 - \phi}{\alpha\lambda}(\epsilon - \beta)}{1 + \beta\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)} \right) \\
\implies g_h &= \bar{g} \cdot \left[1 - \frac{\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)(1 - \gamma + \beta - \epsilon) + \left(1 + \frac{1 - \phi}{\alpha\lambda}\right)(\beta - \epsilon)}{1 + \beta\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)} \right]
\end{aligned} \tag{163}$$

In the case that $\gamma \leq 1$, I get

$$\begin{aligned}
g_{\bar{s}} &= g_{\delta} + g_{\bar{v}} \\
\implies g_{\bar{s}} &= \epsilon\alpha \frac{\lambda\bar{n}}{1 - \phi} - \beta\alpha \frac{\lambda(\bar{n} + g_{\bar{s}})}{1 - \phi} - \beta g_{\bar{s}} + (\epsilon - \beta)\bar{n} \\
\implies g_{\bar{s}} &= \frac{\alpha\lambda\bar{n}}{1 - \phi}(\epsilon - \beta) - \beta g_{\bar{s}}\left(1 + \frac{\alpha\lambda}{1 - \phi}\right) + (\epsilon - \beta)\bar{n} \\
\implies g_{\bar{s}}\left(1 + \beta + \beta \frac{\alpha\lambda}{1 - \phi}\right) &= \frac{\alpha\lambda\bar{n}}{1 - \phi}(\epsilon - \beta) + (\epsilon - \beta)\bar{n} \\
\implies g_{\bar{s}} &= \frac{\bar{n} [\alpha\lambda(\epsilon - \beta) + (1 - \phi)(\epsilon - \beta)]}{(1 + \beta)(1 - \phi) + \beta\alpha\lambda} \\
\implies g_{\bar{s}} &= \frac{-\bar{n} \left[\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)(\beta - \epsilon) \right]}{1 + \beta\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)}
\end{aligned} \tag{164}$$

which given $\beta > \epsilon$ is negative as conjectured.

I can then calculate g_h :

$$\begin{aligned}
 g_h &= \frac{\alpha\lambda\bar{n}}{1-\phi} + \left(1 + \frac{\alpha\lambda}{1-\phi}\right) g_{\bar{s}} \\
 \implies g_h &= \frac{\alpha\lambda\bar{n}}{1-\phi} + \left(1 + \frac{\alpha\lambda}{1-\phi}\right) \frac{\bar{n} \left[\left(1 + \frac{\alpha\lambda}{1-\phi}\right)(\epsilon - \beta) \right]}{1 + \beta\left(1 + \frac{\alpha\lambda}{1-\phi}\right)} \\
 \implies g_h &= \bar{g} \cdot \left[1 - \frac{\left(2 + \frac{\alpha\lambda}{1-\phi} + \frac{1-\phi}{\alpha\lambda}\right)(\beta - \epsilon)}{1 + \beta\left(1 + \frac{\alpha\lambda}{1-\phi}\right)} \right]
 \end{aligned}$$

Finally, note that given $g_{\bar{v}} \geq 0$ and $g_{\bar{s}} < 0$ in both cases, since $g_{\bar{v}} + g_{\delta} = g_{\bar{s}}$, I know g_{δ} is negative, and thus I know that δ_t falls exponentially to zero.

A.5 Proof of Proposition 4

In the case that $\epsilon < \beta$, the proof is straightforward. In particular, in the previous two proofs for the cases that $\gamma > 1$, when I plug in $\gamma = 1 + (\beta - \epsilon) \left(\frac{1-\phi}{\alpha\lambda} + 1 \right)$ it immediately follows that $g_s = 0$.

In the case that $\epsilon = \beta$ and $\gamma \leq 1$, the proof is straightforward as well. In particular, consider the $\gamma \leq 1$ case in the previous proof; plugging in $\epsilon = \beta$ immediately yields $g_s = 0$.

Once $g_s = 0$, the proof proceeds as in the rule of thumb allocation.

A.6 Proof of Proposition 5

This proof is essentially the same as for proposition 2, including the section on the minimum valid value of ρ , with slight modifications.

First, $g_s < 0$ follows directly from $\gamma > 1$ and $\epsilon > \beta$, so no additional condition is necessary.

Second, I need to ensure that $g_c > 0$, which is necessary since I am assuming $c_t \rightarrow \infty$ such that $g_{\bar{v}} \rightarrow 0$. If $\epsilon \gg \beta$, this is not the case. Specifically,

$$\begin{aligned}
 g_c &= \bar{g} \cdot \left[\frac{\beta + (\beta - \epsilon) \frac{1-\phi}{\alpha\lambda}}{\gamma + \epsilon - 1} \right] > 0 \\
 &\iff \beta + (\beta - \epsilon) \frac{1-\phi}{\alpha\lambda} > 0 \\
 &\iff \beta \frac{\alpha\lambda}{1-\phi} > \epsilon - \beta \\
 &\iff \frac{\epsilon - \beta}{\beta} < \frac{\alpha\lambda}{1-\phi}
 \end{aligned} \tag{165}$$

A.7 Proof of Proposition 6

I conjecture that s_t (and l_t) fall exponentially to 0, while consumption grows at a positive rate.

Given $\gamma \leq 1$, $\frac{u(c_t)}{u'(c_t)c_t}$ is asymptotically constant, so

$$\begin{aligned} g_{\tilde{v}} &= g \frac{u(c_t)}{u'(c_t)c_t} - g\rho + \delta_t - g_{vt} \\ \implies g_{\tilde{v}} &= 0 \end{aligned} \tag{166}$$

as long as δ_t converges to a constant.

For s_t (and l_t) to fall exponentially to 0, $\delta_t \tilde{v}_t$ must rise and eventually converges asymptotically to $1/\epsilon$. Therefore, on the balanced growth path, $g_{\tilde{v}} = -g_{\delta}$. Thus, $0 = -g_{\delta}$.

The growth rates $g_{\delta} = \alpha(\epsilon g_A - \beta g_B) + \epsilon g_s + (\epsilon - \beta)\bar{n}$, $g_A = \frac{\lambda(g_s + \bar{n})}{1 - \phi}$, and $g_B = \frac{\lambda\bar{n}}{1 - \phi}$ follow just as in the proof of proposition 2.

Thus,

$$\begin{aligned} 0 &= g_{\delta} = \alpha(\epsilon g_A - \beta g_B) + \epsilon g_s + (\epsilon - \beta)\bar{n} \\ \implies 0 &= \alpha\left(\epsilon \frac{\lambda(g_s + \bar{n})}{1 - \phi} - \beta \frac{\alpha\lambda\bar{n}}{1 - \phi}\right) + \epsilon g_s + (\epsilon - \beta)\bar{n} \\ \implies g_s \cdot (-\epsilon)\left(1 + \frac{\alpha\lambda}{1 - \phi}\right) &= (\epsilon - \beta)\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)\bar{n} \\ \implies g_s &= -\frac{\epsilon - \beta}{\epsilon}\bar{n} \end{aligned} \tag{167}$$

which is negative, as conjectured.

Just like in the proof of proposition 2, $g_h = \alpha g_B = \frac{\alpha\lambda\bar{n}}{1 - \phi} \equiv g$ follows.

I can then calculate g_c :

$$\begin{aligned} g_c &= \alpha g_A + g_s \\ \implies g_c &= \alpha \frac{\lambda(g_s + \bar{n})}{1 - \phi} + g_s \\ \implies g_c &= \bar{n} \cdot \left[\frac{\alpha\lambda}{1 - \phi} - \left(1 + \frac{\alpha\lambda}{1 - \phi}\right) \frac{\epsilon - \beta}{\epsilon} \right] \\ \implies g_c &= \bar{g} - (\bar{n} + \bar{g}) \frac{\epsilon - \beta}{\epsilon} \end{aligned} \tag{168}$$

Given $\epsilon > \beta$, it follows that $g_c < \bar{g}$.

I have to check that $g_c > 0$ (since $g_v = 0$ requires $c_t \rightarrow \infty$).

$$\begin{aligned}
g_c &= \bar{g} - (\bar{n} + \bar{g}) \frac{\epsilon - \beta}{\epsilon} > 0 \\
&\iff \epsilon \frac{\alpha\lambda}{1 - \phi} > (1 + \frac{\alpha\lambda}{1 - \phi})(\epsilon - \beta) \\
&\iff 0 > (\epsilon - \beta) - \beta \frac{\alpha\lambda}{1 - \phi} \\
&\iff \frac{\epsilon - \beta}{\beta} < \frac{\alpha\lambda}{1 - \phi}
\end{aligned} \tag{169}$$

which is the same condition that $\epsilon \gg \beta$ from before.

Finally, I determine $\delta_t \rightarrow \delta^*$. In the case that $\gamma < 1$:

$$\begin{aligned}
\delta^* &= \frac{1}{\epsilon} \lim_{t \rightarrow \infty} 1/\tilde{v}_t \\
\implies \delta^* &= \frac{1}{\epsilon} \lim_{t \rightarrow \infty} \frac{\rho + \delta_t - g_{vt}}{\bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma}} \\
\implies \delta^* &= \frac{(\rho + \delta^* - g_v)(1 - \gamma)}{\epsilon}
\end{aligned}$$

Note that $v_t = \frac{u(c_t)}{\rho + \delta_t - g_{vt}}$, so given a positive and thus asymptotically constant denominator, $g_v = \lim_{t \rightarrow \infty} \frac{u'(c_t)c_t}{u(c_t)} = \lim_{t \rightarrow \infty} \frac{u'(c_t)c}{u(c_t)} \cdot g_c = \lim_{t \rightarrow \infty} 1/(\bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma}) \cdot g_c = (1 - \gamma)g_c$.

$$\begin{aligned}
\implies \delta^* &= \frac{(\rho + \delta^* - (1 - \gamma)g_c)(1 - \gamma)}{\epsilon} \\
\implies \delta^*(1 - \frac{1 - \gamma}{\epsilon}) &= \frac{(1 - \gamma)\rho - (1 - \gamma)^2 g_c}{\epsilon} \\
\implies \delta^* &= \frac{(1 - \gamma)\rho - (1 - \gamma)^2 g_c}{\epsilon(1 - \frac{1 - \gamma}{\epsilon})} \\
\implies \delta^* &= \frac{(1 - \gamma)\rho - (1 - \gamma)^2 g_c}{\epsilon + \gamma - 1}
\end{aligned} \tag{170}$$

Since $\epsilon > 1 - \gamma$, the denominator is positive.

For the numerator to be positive, it must be the case that $(1 - \gamma)\rho - (1 - \gamma)^2 g_c > 0 \iff \rho > (1 - \gamma)g_c$. This ensures that our integral over utility is

bounded: $u(c_t)$ grows at rate $(1 - \gamma)g_c$ asymptotically, so if ρ were smaller than that, the integral could be unbounded. (And we can't solve cases with potential unbounded utility with our mathematical methods.)

Thus, δ does indeed converge to a constant as conjectured.

When $\gamma = 1$, $\frac{u(c_t)}{u'(c_t)c_t} = \bar{u} + \ln(c_t)$, so $\delta_t \rightarrow 0$. However, δ_t does not fall fast enough (not exponentially, $g_\delta = 0$)—in particular, $\delta_t \rightarrow 0$ proportional to $1/\tilde{v}_t$ proportional to $1/\ln(c_t)$ —so $M_t \rightarrow 0$ even when $\gamma = 1$.

A.8 Proof of Proposition 7

I conjecture that the asymptotic growth path features positive consumption growth and $\delta_t \rightarrow \infty$ with constant asymptotic $g_\delta > 0$, with $\sigma_t \rightarrow \sigma^*$. Given $\gamma \leq 1$, $\frac{u(c_t)}{u'(c_t)c_t} \rightarrow \frac{1}{1-\gamma}$ asymptotically.

Recall that

$$\delta_t \tilde{v}_t = \delta_t \frac{u(c_t)/(u'(c_t)c_t)}{\rho + \delta - g_{vt}}.$$

On an asymptotic growth path, $g_{vt} = g_v$. Thus, as δ_t gets large and $\frac{u(c_t)}{u'(c_t)c_t} \rightarrow \frac{1}{1-\gamma}$ on the asymptotic growth path,

$$\delta_t \tilde{v}_t \rightarrow \frac{1}{1-\gamma}. \quad (171)$$

Thus, $\delta_t \tilde{v}_t$ is asymptotically constant. Note that $\delta_t \tilde{v}_t < 1/\epsilon$ must still be satisfied, but since $\epsilon < 1 - \gamma$, $1/\epsilon > 1/(1 - \gamma)$, so this holds.

In turn, $\delta_t \tilde{v}_t$ being asymptotically constant means s_t and ℓ_t converge to constants s^* and ℓ^* . The rest follows as in the rule of thumb allocation; in particular, we get positive asymptotic consumption growth and positive growth of δ_t (since $\epsilon > \beta$), as conjectured.

We can determine ℓ^* . Recall that

$$\begin{aligned} \frac{1 - \ell_t}{\ell_t} &= \frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t} \\ \implies \frac{1 - \ell^*}{\ell^*} &= \frac{\beta \frac{1}{1-\gamma}}{1 - \epsilon \frac{1}{1-\gamma}} \\ \implies \ell^* &= \frac{1}{1 + \frac{\beta \frac{1}{1-\gamma}}{1 - \epsilon \frac{1}{1-\gamma}}} \\ \implies \ell^* &= \frac{1 - \epsilon \frac{1}{1-\gamma}}{1 - \epsilon \frac{1}{1-\gamma} + \beta \frac{1}{1-\gamma}} \\ \implies \ell^* &= \frac{1 - \gamma - \epsilon}{1 - \gamma - \epsilon + \beta} \end{aligned} \quad (172)$$

Given $\epsilon < 1 - \gamma$, ℓ^* is clearly positive and less than one.

A.9 Proof of Proposition 8

I again conjecture that the asymptotic growth path features positive consumption growth g_c and $\delta_t \rightarrow \infty$ with positive g_δ . As in the previous proof, we get $\delta_t \tilde{v}_t \rightarrow \frac{1}{1-\gamma}$. However, since now $\frac{1}{1-\gamma} = \frac{1}{\epsilon}$, $\frac{\beta \delta_t \tilde{v}_t}{1-\epsilon \delta_t \tilde{v}_t} \rightarrow \infty$ asymptotically. Thus, since $(1-\ell_t)/\ell_t$ and $(1-s_t)/s_t$ are both proportional to $\frac{\beta \delta_t \tilde{v}_t}{1-\epsilon \delta_t \tilde{v}_t}$ on an asymptotic growth path, ℓ_t and s_t go to 0. I conjecture that ℓ_t and s_t fall to zero at constant exponential rates, while $\sigma_t \rightarrow \sigma^*$.

We can plug in the given $\epsilon = 1 - \gamma$:

$$\begin{aligned} 1 - \epsilon \delta_t \tilde{v}_t &= 1 - (1 - \gamma) \delta_t \frac{\bar{u} c_t^{\gamma-1} + \frac{1}{1-\gamma}}{\rho + \delta_t - g_{vt}} \\ \implies 1 - \epsilon \delta_t \tilde{v}_t &= 1 - \left((1 - \gamma) \delta_t \frac{\bar{u} c_t^{\gamma-1}}{\rho + \delta_t - g_{vt}} + \frac{\delta_t}{\rho + \delta_t - g_{vt}} \right) \end{aligned}$$

Thus, asymptotically, as $\delta_t \rightarrow \infty$ and $g_{vt} = g_v$,

$$\begin{aligned} \implies 1 - \epsilon \delta_t \tilde{v}_t &\rightarrow 1 - ((1 - \gamma) \bar{u} c_t^{\gamma-1} + 1) \\ \implies 1 - \epsilon \delta_t \tilde{v}_t &\rightarrow -(1 - \gamma) \bar{u} c_t^{\gamma-1}. \end{aligned} \tag{173}$$

(Note that the RHS is still positive, as we are assuming \bar{u} negative when $\gamma < 1$.)

In particular, note that $\beta \delta_t \tilde{v}_t$ is constant asymptotically, as is $(1-s_t)$ and $(1-\ell_t)$. Thus, from the FOCs we get that $g_s = g_\ell = g_{1-\epsilon \delta_t \tilde{v}_t}$.

$$\begin{aligned} g_s = g_\ell = g_{1-\epsilon \tilde{v}_t} &= -(1 - \gamma) g_c \\ \implies g_s = g_\ell &= -\epsilon g_c. \end{aligned} \tag{174}$$

g_s and g_ℓ are indeed negative as conjectured.

As in proposition 2, we have

$$g_c = \alpha g_A + g_s \tag{175}$$

$$g_A = \frac{\lambda(g_s + \bar{n})}{1 - \phi} \tag{176}$$

$$g_h = \alpha g_B \tag{177}$$

$$g_B = \frac{\lambda \bar{n}}{1 - \phi} \tag{178}$$

$$g_\delta = (\epsilon - \beta) \bar{n} + \epsilon g_c - \beta g_h \tag{179}$$

(Verifying $\sigma_t \rightarrow \sigma^*$ proceeds similarly as in that proposition as well.)

We can use this to solve for an explicit expression for g_c

$$\begin{aligned}
 g_c &= \alpha \frac{\lambda(-\epsilon g_c + \bar{n})}{1 - \phi} - \epsilon g_c \\
 1 + \epsilon \left(1 + \frac{\alpha \lambda}{1 - \phi} \right) g_c &= \frac{\alpha \lambda}{1 - \phi} \bar{n} \\
 g_c &= \frac{\frac{\alpha \lambda}{1 - \phi}}{1 + \epsilon \left(1 + \frac{\alpha \lambda}{1 - \phi} \right)} \bar{n} = \frac{1}{1 + \epsilon \left(1 + \frac{\alpha \lambda}{1 - \phi} \right)} \bar{g} < \bar{g}, \quad (180)
 \end{aligned}$$

which is indeed positive, as conjectured.

From this we get an explicit expression for g_s as well:

$$g_s = g_\ell = -\epsilon \frac{\frac{\alpha \lambda}{1 - \phi}}{1 + \epsilon \left(1 + \frac{\alpha \lambda}{1 - \phi} \right)} \bar{n}.$$

We finally must verify that $g_\delta > 0$ as conjectured.

$$\begin{aligned}
 g_\delta &= (\epsilon - \beta) \bar{n} + \epsilon g_c - \beta g_h \\
 \implies g_\delta &= (\epsilon - \beta) \bar{n} + \left(\frac{\epsilon}{1 + \epsilon \left(1 + \frac{\bar{g}}{\bar{n}} \right)} - \beta \right) \bar{g}
 \end{aligned}$$

From this, we get that $g_\delta > 0$ iff

$$\frac{\alpha \lambda}{1 - \phi} < \frac{(\epsilon - \beta)(1 + \epsilon)}{\beta \epsilon}.$$

A.10 Proof of Proposition 9

First, note that the asymptotic growth path must feature $g_c > 0$.

To see this, note first that $g_c < 0$ is not permissible, since then consumption would eventually fall below the zero utility level. If $g_c = 0$ were the case, that would require $g_s = g_\ell < 0$. But $g_c = 0$ would also mean $g_\delta < 0$. This is because given our condition that $\frac{\alpha\lambda}{1-\phi} \geq \frac{(\epsilon-\beta)(1+\epsilon)}{\beta\epsilon}$, it must also be true that $\frac{\alpha\lambda}{1-\phi} > \frac{(\epsilon-\beta)}{\beta}$, so $\beta\bar{g} > (\epsilon - \beta)\bar{n}$. Thus, $g_\delta = (\epsilon - \beta)\bar{n} - \beta g_h + \epsilon g_c = (\epsilon - \beta)\bar{n} - \beta\bar{g} < 0$. But that would mean $\delta_t \rightarrow 0$, so $\delta_t \tilde{v}_t \rightarrow 0$ (since \tilde{u} is bounded), and thus s_t and ℓ_t would go to 1, contradicting $g_s = g_\ell < 0$.

Moreover, I claim that it must be the case that $g_\delta = 0$.

If $g_\delta > 0$, we get the result as in the proof of proposition 8, where $g_c > 0$ only if $\frac{\alpha\lambda}{1-\phi} < \frac{(\epsilon-\beta)(1+\epsilon)}{\beta\epsilon}$, which is not met here.

If $g_\delta < 0$, $\delta_t \tilde{v}_t \rightarrow 0$, so s_t and ℓ_t to 1. But then consumption grows at least as fast as safety, and so given $\epsilon > \beta$, $g_\delta > 0$.

Thus, it must be the case that $g_\delta = 0$. However, it also can't be the case that δ is bounded. If that were to case, $\delta_t \tilde{v}_t$ would be bounded by something strictly less than $1/\epsilon$. In turn, s_t and ℓ_t could not sustain constant negative exponential growth rates. But then we would get balanced growth, with $g_\delta > 0$ since $\epsilon > \beta$, which is a contradiction.

Thus, it must be the case that $\delta \rightarrow \infty$ subexponentially. Now, since δ_t still goes to ∞ , $\delta_t \tilde{v}_t \rightarrow \tilde{u} = \frac{1}{1-\gamma} = \frac{1}{\epsilon}$, so s_t and ℓ_t go to zero.

In particular, if $g_s = g_\ell$ were equal to zero, we would get balanced growth with $g_\delta > 0$ (given $\epsilon > \beta$). Thus, it must be the case that $g_s = g_\ell < 0$. (They can't be > 0 , since s_t and ℓ_t are going to zero.)

As in the proof for proposition 6, for $g_\delta = 0$ to be the case when s_t and ℓ_t fall exponentially to 0, it must be the case that

$$g_\ell = g_s = -\frac{\epsilon - \beta}{\epsilon}\bar{n}, \quad (181)$$

and the other growth rates follow as in that proof, in particular $g_c = \bar{g} - (\bar{n} + \bar{g})\frac{\epsilon-\beta}{\epsilon}$. $g_c > 0$ holds when $\frac{\alpha\lambda}{1-\phi} > \frac{(\epsilon-\beta)}{\beta}$ as shown in that proof. As we show earlier in this proof, this condition hold in this case.

Note that now $\delta_t \tilde{v}_t$'s approach to $1/\epsilon$ is driven not by the growth in c_t but rather by the subexponential growth in δ_t and thus the approach of $\frac{\delta_t}{\rho + \delta_t - g_v}$ to 1. Recall that $\delta_t \tilde{v}_t \rightarrow \frac{\delta_t}{\rho + \delta_t - g_v} \tilde{u}_t$. c_t grows much quicker than δ_t (exponential vs. subexponential), so \tilde{u} is asymptotically $\frac{1}{1-\gamma}$ quicker than

$\frac{\delta_t}{\rho + \delta_t - g_v}$ is asymptotically 1. (Thus, the relationship $g_s = -\epsilon g_c$ derived in the proof of proposition 8 does not apply here.)

A.11 Proof of Proposition 10

First, note that the optimal allocation cannot feature an asymptotic growth path with $g_c < 0$. This would mean $c_t \rightarrow 0$, but eventually c_t would fall below the zero utility level, i.e. dying would be preferred to living.

We also cannot have $g_c > 0$ on an asymptotic growth path.

To see this, note that $\delta_t \rightarrow \infty$ no matter what. Moreover, that δ_t can grow at most exponentially on an asymptotic path because \bar{n} grows exponentially and c_t can grow at most exponentially. Thus, $g_{vt} = g_v$, from which follows

$$\delta_t \tilde{v}_t = \delta_t \frac{\tilde{u}}{\rho + \delta_t - g_{vt}} \rightarrow \tilde{u}$$

as $\delta_t \rightarrow \infty$.

Now, we consider three different cases of γ separately. If $\gamma > 1$, $\bar{u} = \bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma} \rightarrow \infty$ as $c_t \rightarrow \infty$ as we saw previously. But then $\delta_t \tilde{v}_t$ goes above the bound of $1/\epsilon$ we get from our FOCs. Thus, c_t can't grow indefinitely and must be bounded above.

If $1 - \epsilon < \gamma < 1$, we have $\tilde{u} = \bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma} \rightarrow \frac{1}{1-\gamma}$. But $\frac{1}{1-\gamma} > \frac{1}{\epsilon}$, so again we would have $\delta_t \tilde{v}_t$ go above the bound of $1/\epsilon$. Again, c_t can't grow indefinitely and must be bounded above.

Finally, consider $\gamma = 1$, i.e. log utility. Then, $\tilde{u} = \frac{u(c_t)}{u'(c_t)c_t} = \frac{\bar{u} + \log(c_t)}{1/c_t \cdot c_t} = \bar{u} + \log(c_t) = u(c_t)$. Thus, as $c_t \rightarrow \infty$, \bar{u} grows without bound, and $\delta_t \tilde{v}_t$ goes above the bound of $1/\epsilon$. Again, c_t can't grow indefinitely and must be bounded above.

Thus, we must have $g_c = 0$. For this to be the case on an asymptotic growth path (with $\sigma_t \rightarrow \sigma^*$), we must have

$$\begin{aligned} 0 = g_c &= \alpha \frac{\lambda(g_s + \bar{n})}{1 - \phi} + g_s \\ \implies \left(1 + \frac{\alpha\lambda}{1 - \phi}\right)g_s &= -\frac{\alpha\lambda}{1 - \phi}\bar{n} \\ \implies g_s &= -\frac{1}{\left(1 + \frac{\alpha\lambda}{1 - \phi}\right)}\bar{g}, \end{aligned} \tag{182}$$

so s_t must fall to zero at a constant exponential rate. Note that on an asymptotic growth path $g_\ell = g_s$ by our FOCs, so ℓ_t falls to 0 as well. The other growth rates then follow as usual.

Since s_t and ℓ_t fall to zero, $\delta_t \tilde{v}_t$ must converge to $1/\epsilon$. But $\delta_t \tilde{v}_t \rightarrow \tilde{u}$. Thus, c_t must increase and converge to a level c^* such that $\tilde{u} \rightarrow 1/\epsilon$. In particular, this means that c^* is given as follows:

$$\tilde{u} = \frac{u(c^*)}{u'(c^*)c^*} = \frac{1}{\epsilon}. \quad (183)$$

We now differentiate based on γ . If $\gamma = 1$, this means

$$\begin{aligned} \bar{u} + \log(c^*) &= \frac{1}{\epsilon} \\ \iff c^* &= \exp\left(\frac{1}{\epsilon} - \bar{u}\right). \end{aligned} \quad (184)$$

The zero utility level c_0 is the level of consumption at which $\log(c_0) = -\bar{u} \iff c_0 = \exp(-\bar{u})$, so

$$\frac{c^*}{c_0} = \exp\left(\frac{1}{\epsilon}\right). \quad (185)$$

If $\gamma > 1$ or $1 > \gamma > 1 - \epsilon$,

$$\begin{aligned} \bar{u}c^{*\gamma-1} + \frac{1}{1-\gamma} &= \frac{1}{\epsilon} \\ \iff c^* &= \left[\frac{\frac{1}{\epsilon} + \frac{1}{\gamma-1}}{\bar{u}} \right]^{\frac{1}{\gamma-1}} \end{aligned} \quad (186)$$

(Note that if $\gamma > 1$, we are assuming $\bar{u} > 0$, so this is positive; if $\gamma < 1$, we are assuming $\bar{u} < 0$, so this is positive then too.)

We determine c_0 :

$$\begin{aligned} \bar{u} &= -\frac{c_0^{1-\gamma}}{1-\gamma} \\ \implies c_0 &= \left[\frac{1}{(\gamma-1)\bar{u}} \right]^{\frac{1}{\gamma-1}} \end{aligned} \quad (187)$$

Then,

$$\begin{aligned} \frac{c^*}{c_0} &= \left[\frac{(\frac{1}{\epsilon} + \frac{1}{\gamma-1})(\gamma-1)\bar{u}}{\bar{u}} \right]^{\frac{1}{\gamma-1}} \\ \implies \frac{c^*}{c_0} &= \left[\frac{\gamma-1}{\epsilon} + 1 \right]^{\frac{1}{\gamma-1}}. \end{aligned} \quad (188)$$

When $\gamma > 1$, this is clearly positive; when $1 > \gamma > 1 - \epsilon$, $1 - \gamma < \epsilon$, so $\frac{\gamma-1}{\epsilon} < 1$, so this is positive as well.

A.12 Proof of Proposition 11

Note that with this knife-edge condition, if $g_c = 0$ and g_h is at “full speed”, i.e. $g_h = \bar{g}$, we have $g_\delta = 0$. If $g_c > 0$, $g_\delta > 0$.

Again, we can't have $g_c < 0$ on the asymptotic path in the optimal allocation, because consumption would fall below the zero utility level.

Similarly, we can't have $g_c > 0$, since then $g_\delta > 0$, so $\delta_t \rightarrow \infty$, $\delta_t \tilde{v}_t \rightarrow \tilde{u}$, and \tilde{u} becomes bigger than $1/\epsilon$ eventually (in all cases, whether $\gamma > 1$, $\gamma = 1$, or $1 > \gamma > 1 - \epsilon$).

Thus, the asymptotic growth path must feature $g_c = 0$. In turn, this means $g_\delta = 0$.

$g_c = 0$, requires, as in proposition 10, that

$$g_\ell = g_s = -\frac{1}{\left(1 + \frac{\alpha\lambda}{1-\phi}\right)}\bar{g}. \quad (189)$$

The other growth rates follow as usual.

In the case that $\gamma \geq 1$, c_t must be bounded. To see this, note that if c_t increases (subexponentially) without bound, $\tilde{u} \rightarrow \infty$, so we would need $\delta_t \rightarrow 0$ to keep $\delta_t \tilde{v}_t < 1/\epsilon$. But this is impossible, since we can at most keep the growth in δ_t due to the scale effect in check, not decrease δ_t to zero.

In the case that $\gamma < 1$, c_t can increase without bound, since \tilde{u} is still bounded. δ_t must then converge to a level δ^* such that $\delta^* \frac{1}{1-\gamma} = \frac{1}{\epsilon}$, i.e. $\delta^* = \frac{1-\gamma}{\epsilon} < 1$.

A.13 Proof of Proposition 12

Note that for any variable a , $\widehat{(1-a)} = \frac{1-\dot{a}}{1-a} = -\frac{\dot{a}}{a} \frac{a}{1-a} = -\hat{a} \frac{a}{1-a}$.

Law of Motion: y

Recall that $y \equiv g_{At} = \frac{(s_t \sigma_t N_t)^\lambda}{A_t^{1-\phi}}$. Taking logs and derivatives:

$$\hat{y} = \lambda(\bar{n} + \hat{\sigma} + \hat{s}) - (1 - \phi)y \quad (190)$$

Law of Motion: z

Similarly, I consider \hat{z} . Recall that $z \equiv g_{Bt} = \frac{((1-s_t)\sigma_t N_t)^\lambda}{B_t^{1-\phi}}$. Taking logs and derivatives:

$$\hat{z} = \lambda \left(\bar{n} + \hat{\sigma} - \hat{s} \frac{s}{1-s} \right) - (1 - \phi)z \quad (191)$$

Law of Motion: δ

Recall that $\delta_t = \bar{\delta} N_t^{\epsilon-\beta} c_t^\epsilon h_t^{-\beta}$, with $c_t = A_t^\alpha \ell_t (1 - \sigma_t)$ and $B_t^\alpha (1 - \ell_t) (1 - \sigma_t)$. Again, taking logs and derivatives:

$$\begin{aligned} \hat{\delta} &= (\epsilon - \beta)\bar{n} + \epsilon g_{ct} - \beta g_{ht} \\ \implies \hat{\delta} &= (\epsilon - \beta)\bar{n} + \epsilon \left(\alpha y + \hat{\ell} - \hat{\sigma} \frac{\sigma}{1-\sigma} \right) - \beta \left(\alpha z - \hat{\ell} \frac{\ell}{1-\ell} - \hat{\sigma} \frac{\sigma}{1-\sigma} \right) \\ \implies \hat{\delta} &= (\epsilon - \beta) \left(\bar{n} - \hat{\sigma} \frac{\sigma}{1-\sigma} \right) + \alpha(\epsilon y - \beta z) + \hat{\ell} \left(\epsilon + \beta \frac{\ell}{1-\ell} \right) \end{aligned} \quad (192)$$

Law of Motion: s

Next, I consider \hat{s} . Recall the FOC for s_t : $\frac{1-s_t}{s_t} = \frac{p_{bt} \dot{B}_t}{p_{at} \dot{A}_t} = \frac{p_{bt} ((1-s_t)\sigma_t N_t)^\lambda B_t^\phi}{p_{at} (s_t \sigma_t N_t)^\lambda A_t^\phi}$. Taking logs and derivatives of both sides:

$$\begin{aligned} -\hat{s} \frac{s}{1-s} - \hat{s} &= g_{p_{bt}} + \lambda \left(-\hat{s} \frac{s}{1-s} + \hat{\sigma} + \bar{n} \right) + \phi z - g_{p_{at}} - \lambda(\hat{s} + \hat{\sigma} + \bar{n}) - \phi y \\ \implies \hat{s} \left(1 + \frac{s}{1-s} \right) &= g_{p_{at}} - g_{p_{bt}} + \phi y - \phi z + \lambda \hat{s} \left(1 + \frac{s}{1-s} \right) \\ \implies \hat{s} \frac{1-\lambda}{1-s} &= g_{p_{at}} - g_{p_{bt}} + \phi y - \phi z \end{aligned} \quad (193)$$

Recall FOCs for A_t and B_t : $\frac{\dot{p}_{at}}{p_{at}} = \rho - \frac{1}{p_{at}} [M_t u'(c_t) \alpha \frac{c_t}{A_t} + p_{at} \phi \frac{\dot{A}_t}{A_t} - \alpha \epsilon v_t M_t \frac{\delta_t}{A_t}]$ and $\frac{\dot{p}_{bt}}{p_{bt}} = \rho - \frac{1}{p_{bt}} [p_{bt} \phi \frac{\dot{B}_t}{B_t} + \alpha \beta v_t M_t \frac{\delta_t}{B_t}]$ respectively. Substituting, I get

$$\begin{aligned} \hat{s} \frac{1-\lambda}{1-s} &= \rho - \frac{1}{p_{at}} [M_t u'(c_t) \alpha \frac{c_t}{A_t} + p_{at} \phi \frac{\dot{A}_t}{A_t} - \alpha \epsilon v_t M_t \frac{\delta_t}{A_t}] \\ &\quad - \rho + \frac{1}{p_{bt}} [p_{bt} \phi \frac{\dot{B}_t}{B_t} + \alpha \beta v_t M_t \frac{\delta_t}{B_t}] + \phi y - \phi z \\ \implies \hat{s} &= \frac{1-s}{1-\lambda} \left[\frac{\alpha M_t \beta \delta_t v_t}{p_{bt} B_t} - \frac{\alpha M_t (u'(c_t) c_t - \epsilon \delta_t v_t)}{p_{at} A_t} \right] \end{aligned} \quad (194)$$

Recall the FOC for σ_t : $\frac{1-\sigma_t}{\sigma_t} = \frac{M_t [u'(c_t) c_t + (\beta - \epsilon) \delta_t v_t]}{\lambda (p_{at} \dot{A} + p_{bt} B)}$. From the FOC for s_t , I know $p_{bt} \dot{B}_t = \frac{1-s_t}{s_t} p_{at} \dot{A}_t$; substituting this yields:

$$\begin{aligned} \lambda \frac{1-\sigma_t}{\sigma_t} &= \frac{M_t [u'(c_t) c_t + (\beta - \epsilon) \delta_t v_t]}{p_{at} \dot{A} \left(1 + \frac{1-s_t}{s_t}\right)} \\ \implies \lambda \frac{y}{s_t} \frac{1-\sigma_t}{\sigma_t} &= \frac{M_t [u'(c_t) c_t + (\beta - \epsilon) \delta_t v_t]}{p_{at} A_t} \end{aligned} \quad (195)$$

Similarly, I get:

$$\lambda \frac{z}{1-s_t} \frac{1-\sigma_t}{\sigma_t} = \frac{M_t [u'(c_t) c_t + (\beta - \epsilon) \delta_t v_t]}{p_{bt} B_t} \quad (196)$$

Now, recall the FOC for ℓ_t : $\frac{1-\ell_t}{\ell_t} = \frac{\beta \delta_t \tilde{v}_t}{1-\epsilon \delta_t \tilde{v}_t}$. I manipulate this:

$$\begin{aligned} 1 - \ell_t - \epsilon \delta_t \tilde{v}_t + \ell_t \epsilon \delta_t \tilde{v}_t &= \ell_t \beta \delta_t \tilde{v}_t \\ \implies \ell_t (\beta - \epsilon) \delta_t v_t &= (1 - \ell_t) u'(c_t) c_t - \epsilon \delta_t v_t \end{aligned} \quad (197)$$

$$\implies (1 - \ell_t) u'(c_t) c_t = \epsilon \delta_t v_t + \ell_t (\beta - \epsilon) \delta_t v_t \quad (198)$$

Combining (195) and (197) gives us

$$\lambda \frac{y}{s_t} \ell \frac{1-\sigma_t}{\sigma_t} = \frac{M_t [u'(c_t) c_t - \epsilon \delta_t v_t]}{p_{at} A_t} \quad (199)$$

Similarly, combining (196) and (198) gives us

$$\lambda \frac{z}{1-s_t} (1-\ell) \frac{1-\sigma_t}{\sigma_t} = \frac{M_t \beta \delta_t v_t}{p_{bt} B_t} \quad (200)$$

Substituting (199) and (200) into (194) yields:

$$\begin{aligned}\hat{s} &= \frac{1-s}{1-\lambda} \left[\alpha \lambda \frac{z}{1-s_t} (1-\ell) \frac{1-\sigma_t}{\sigma_t} - \alpha \frac{y}{s_t} \ell \frac{1-\sigma_t}{\sigma_t} \right] \\ \implies \hat{s} &= \alpha z \frac{\lambda}{1-\lambda} (1-\ell) \frac{1-\sigma}{\sigma} - \alpha y \frac{\lambda}{1-\lambda} \frac{1-s}{s} \ell \frac{1-\sigma}{\sigma}\end{aligned}\quad (201)$$

Law of Motion: σ

I can use the FOC for σ_t , (198), (196) and rearrange:

$$\begin{aligned}\frac{1-\sigma_t}{\sigma_t} &= \frac{M_t[u'(c_t)c_t + (\beta - \epsilon)\delta_t v_t]}{\lambda(p_{at}\dot{A} + p_{bt}\dot{B})} \\ \implies \frac{1-\sigma_t}{\sigma_t} &= \frac{1-\ell}{1-\ell} \frac{M_t[u'(c_t)c_t + (\beta - \epsilon)\delta_t v_t]}{\lambda \left(1 + \frac{s_t}{1-s_t}\right) (p_{bt}\dot{B})} \\ \implies \frac{1-\sigma_t}{\sigma_t} &= \frac{1-s_t}{1-\ell} \frac{M_t[\beta\delta_t v_t]}{\lambda(p_{bt}g_{Bt}B_t)}\end{aligned}\quad (202)$$

Taking logs and derivatives yields:

$$-\hat{\sigma} \frac{\sigma}{1-\sigma} - \hat{\sigma} = -\hat{s} \frac{s}{1-s} + \hat{\ell} \frac{\ell}{1-\ell} - \delta + \hat{\delta} + g_{vt} - g_{p_{bt}} - \hat{z} - z \quad (203)$$

From the FOC for M_t , I get $g_{vt} = \frac{\dot{v}_t}{v_t} = \rho - \frac{u(c_t)}{v_t} + \delta_t$. From the FOC for B_t I get $g_{p_{bt}} = \frac{\dot{p}_{bt}}{p_{bt}} = \rho - \frac{1}{p_{bt}} [p_{bt}\phi \frac{\dot{B}_t}{B_t} + \alpha\beta v_t M_t \frac{\delta_t}{B_t}]$. I can substitute in these expressions and the expressions I previously found for $\hat{\delta}$ and \hat{z} . This yields:

$$\begin{aligned}-\hat{\sigma} \frac{\sigma}{1-\sigma} - \hat{\sigma} &= -\hat{s} \frac{s}{1-s} + \hat{\ell} \frac{\ell}{1-\ell} - \delta + \hat{\delta} + \rho - \frac{u(c_t)}{v_t} + \delta_t - \rho + \phi z + \frac{\alpha M_t \beta \delta_t v_t}{p_{bt} B_t} - \hat{z} - z \\ \implies -\hat{\sigma} \frac{1}{1-\sigma} &= -\hat{s} \frac{s}{1-s} + \hat{\ell} \frac{\ell}{1-\ell} + (\epsilon - \beta) \left(\bar{n} - \hat{\sigma} \frac{\sigma}{1-\sigma} \right) + \alpha(\epsilon y - \beta z) + \hat{\ell} \left(\epsilon + \beta \frac{\ell}{1-\ell} \right) \\ &\quad - \frac{u(c_t)}{v_t} + (\phi - 1)z - \lambda \left(\bar{n} + \hat{\sigma} - \hat{s} \frac{s}{1-s} \right) + (1-\phi)z + \frac{\alpha M_t \beta \delta_t v_t}{p_{bt} B_t} \\ \implies -\hat{\sigma} \left(\frac{1}{1-\sigma} + (\beta - \epsilon) \frac{\sigma}{1-\sigma} - \lambda \right) &= -\hat{s} \frac{s}{1-s} (1-\lambda) + \hat{\ell} \left(\frac{\ell}{1-\ell} + \epsilon + \beta \frac{\ell}{1-\ell} \right) \\ &\quad + \bar{n}(\epsilon - \beta - \lambda) + \alpha \epsilon y - \alpha \beta z - \frac{u(c_t)}{v_t} + \frac{\alpha M_t \beta \delta_t v_t}{p_{bt} B_t}\end{aligned}\quad (204)$$

We can plug in (200) and rearrange:

$$\begin{aligned} \implies \hat{\sigma} \frac{1 + (\beta - \epsilon)\sigma - \lambda(1 - \sigma)}{1 - \sigma} &= (1 - \lambda) \frac{s}{1 - s} \hat{s} + (\lambda + \beta - \epsilon) \bar{n} + \alpha\beta z - \alpha\epsilon y + \frac{u(c_t)}{v_t} \\ &+ \left(- \left(\frac{\ell}{1 - \ell} (1 + \beta) + \epsilon \right) \right) \hat{\ell} - \alpha\lambda z \frac{1 - \ell}{1 - s_t} \frac{1 - \sigma_t}{\sigma_t} \end{aligned}$$

I set the following definitions:

$$\theta_\sigma = \frac{1 - \sigma}{1 + (\beta - \epsilon)\sigma - \lambda(1 - \sigma)} \quad (205)$$

$$\omega_\sigma = - \left(\frac{\ell}{1 - \ell} (1 + \beta) + \epsilon \right) \quad (206)$$

$$\mathbb{B} = (1 - \lambda) \frac{s}{1 - s} \hat{s} + (\lambda + \beta - \epsilon) \bar{n} + \alpha\beta z - \alpha\epsilon y + \frac{u(c_t)}{v_t} - \alpha\lambda z \frac{1 - \ell}{1 - s_t} \frac{1 - \sigma_t}{\sigma_t} \quad (207)$$

Then:

$$\hat{\sigma} = \theta_\sigma (\mathbb{B} + \omega_\sigma \hat{\ell}) \quad (208)$$

Law of Motion: ℓ

Recall the FOC for ℓ_t : $\frac{1 - \ell_t}{\ell_t} = \frac{\beta \delta_t \frac{v_t}{u'(c_t)c_t}}{1 - \epsilon \delta_t \frac{v_t}{u'(c_t)c_t}}$. Since $u'(c_t)c_t = c^{1-\gamma}$, taking logs and derivatives, this yields:

$$-\hat{\ell} \frac{1}{1 - \ell} = \hat{\delta} + g_{vt} - (1 - \gamma)g_{ct} + \frac{\epsilon \delta_t \frac{v_t}{u'(c_t)c_t}}{1 - \epsilon \delta_t \frac{v_t}{u'(c_t)c_t}} \left(\hat{\delta} + g_{vt} - (1 - \gamma)g_{ct} \right)$$

Substituting $\frac{1 - \ell_t}{\ell_t} \frac{\epsilon}{\beta} = \frac{\epsilon \delta_t \frac{v_t}{u'(c_t)c_t}}{1 - \epsilon \delta_t \frac{v_t}{u'(c_t)c_t}}$ I get:

$$\hat{\ell} \frac{1}{1 - \ell} = - \left(1 + \frac{1 - \ell}{\ell} \frac{\epsilon}{\beta} \right) (\hat{\delta} + g_{vt} + (\gamma - 1)g_{ct}) \quad (209)$$

From the FOC for M_t , I get $g_{vt} = \frac{\dot{v}_t}{v_t} = \rho - \frac{u(c_t)}{v_t} + \delta_t$. It follows directly

that $g_{ct} = \alpha y + \hat{\ell} - \hat{\sigma} \frac{\sigma}{1-\sigma}$. I can substitute in these two expressions and $\hat{\delta}$:

$$\begin{aligned}
\hat{\ell} &= -(1-\ell) \left(1 + \frac{1-\ell}{\ell} \frac{\epsilon}{\beta} \right) \\
& [(\epsilon - \beta) \left(\bar{n} - \hat{\sigma} \frac{\sigma}{1-\sigma} \right) + \alpha(\epsilon y - \beta z) \\
& + \hat{\ell} \left(\epsilon + \beta \frac{\ell}{1-\ell} \right) + \rho - \frac{u(c_t)}{v_t} + \delta_t + (\gamma - 1) \left(\alpha y + \hat{\ell} - \hat{\sigma} \frac{\sigma}{1-\sigma} \right)] \\
\implies \hat{\ell} & \left[1 + (\gamma - 1 + \epsilon + \beta \frac{\ell}{1-\ell})(1-\ell) \left(1 + \frac{1-\ell}{\ell} \frac{\epsilon}{\beta} \right) \right] \\
& = (1-\ell) \left(1 + \frac{1-\ell}{\ell} \frac{\epsilon}{\beta} \right) \cdot [(\gamma - 1 + \epsilon - \beta) \frac{\sigma}{1-\sigma} \hat{\sigma} + (\beta - \epsilon) \bar{n} \\
& + (1 - \gamma - \epsilon) \alpha y + \alpha \beta z - \rho - \delta + \frac{u(c_t)}{v_t}] \tag{210}
\end{aligned}$$

I set the following definitions:

$$\theta_\ell = \frac{(1-\ell) \left(1 + \frac{1-\ell}{\ell} \frac{\epsilon}{\beta} \right)}{1 + (\gamma - 1 + \epsilon + \beta \frac{\ell}{1-\ell})(1-\ell) \left(1 + \frac{1-\ell}{\ell} \frac{\epsilon}{\beta} \right)} \tag{211}$$

$$\omega_\ell = (\gamma - 1 + \epsilon - \beta) \frac{\sigma}{1-\sigma} \tag{212}$$

$$\mathbb{A} = (\beta - \epsilon) \bar{n} + (1 - \gamma - \epsilon) \alpha y + \alpha \beta z - \rho - \delta + \frac{u(c_t)}{v_t} \tag{213}$$

Then:

$$\hat{\ell} = \theta_\ell (\mathbb{A} + \omega_\ell \hat{\sigma}) \tag{214}$$

I substitute in (208):

$$\begin{aligned}
\hat{\ell} &= \theta_\ell (\mathbb{A} + \omega_\ell \theta_\sigma (\mathbb{B} + \omega_\sigma \hat{\ell})) \\
\implies \hat{\ell} [1 - \omega_\ell \omega_\sigma \theta_\sigma] &= \theta_\ell (\mathbb{A} + \omega_\ell \theta_\sigma \mathbb{B}) \\
\implies \hat{\ell} &= \frac{\theta_\ell (\mathbb{A} + \omega_\ell \theta_\sigma \mathbb{B})}{1 - \omega_\ell \omega_\sigma \theta_\sigma} \tag{215}
\end{aligned}$$

Addendum: $\frac{u(c_t)}{v_t}$

To determine both \textcircled{A} and \textcircled{B} , I need an expression for $\frac{u(c_t)}{v_t}$. First, recall the FOC for ℓ_t : $\frac{1-\ell_t}{\ell_t} = \frac{\beta\delta_t \frac{v_t}{u'(c_t)c_t}}{1-\epsilon\delta_t \frac{v_t}{u'(c_t)c_t}}$. Thus,

$$\begin{aligned} \frac{\ell}{1-\ell} \beta\delta \frac{u(c_t)}{u'(c_t)c_t} &= \frac{u(c_t)}{v_t} \left(1 - \epsilon\delta \frac{v_t}{u'(c_t)c_t}\right) \\ \implies \frac{\ell}{1-\ell} \beta\delta \frac{u(c_t)}{u'(c_t)c_t} &= \frac{u(c_t)}{v_t} - \epsilon\delta \frac{u(c_t)}{u'(c_t)c_t} \\ \implies \frac{\ell}{1-\ell} \beta\delta \frac{u(c_t)}{u'(c_t)c_t} + \epsilon\delta \frac{u(c_t)}{u'(c_t)c_t} &= \frac{u(c_t)}{v_t} \end{aligned} \quad (216)$$

Let $\tilde{u} = \frac{u(c)}{u'(c)c}$. Then,

$$\tilde{u} = \frac{u(c_t)}{u'(c_t)c_t} = \bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma} \quad (217)$$

Thus, I need an expression for c_t . Look at δ :

$$\begin{aligned} \delta_t &= \bar{\delta} N_t^{\epsilon-\beta} c_t^\epsilon h_t^{-\beta} \\ \implies \frac{\delta_t}{\bar{\delta}} \left(\frac{c_t}{h_t}\right)^{-\beta} &= N_t^{\epsilon-\beta} c_t^\epsilon h_t^{-\beta} \frac{c_t^{-\beta}}{h_t^{-\beta}} \\ \implies \left(\frac{\delta_t}{\bar{\delta}} \left(\frac{c_t}{h_t}\right)^{-\beta}\right)^{\frac{1}{\epsilon-\beta}} &= N_t c_t \\ \implies \frac{\left(\frac{\delta_t}{\bar{\delta}} \left(\frac{c_t}{h_t}\right)^{-\beta}\right)^{\frac{1}{\epsilon-\beta}}}{N_t} &= c_t \end{aligned} \quad (218)$$

Next, note that

$$\begin{aligned} \frac{c_t}{h_t} &= \frac{A^\alpha \ell_t (1-\sigma_t)}{B^\alpha (1-\ell_t)(1-\sigma_t)} \\ \implies \frac{c_t}{h_t} &= \frac{\ell_t}{1-\ell_t} \left(\frac{A}{B}\right)^\alpha \end{aligned}$$

Finally, note that

$$\begin{aligned}
\frac{z}{y} &= \frac{\dot{B}_t A_t}{B_t \dot{A}_t} = \frac{A_t (1-s_t)^\lambda \sigma_t^\lambda N_t^\lambda B_t^\phi}{B_t s_t^\lambda \sigma_t^\lambda N_t^\lambda A_t^\phi} \\
\implies \frac{z}{y} &= \frac{A_t^{1-\phi}}{B_t^{1-\phi}} \left(\frac{1-s_t}{s_t} \right)^\lambda \\
\implies \left(\frac{z}{y} \right)^{\frac{1}{1-\phi}} &= \frac{A_t}{B_t} \left(\frac{1-s_t}{s_t} \right)^{\frac{\lambda}{1-\phi}} \\
\implies \left(\frac{z}{y} \right)^{\frac{1}{1-\phi}} &= \frac{A_t}{B_t} \left(\frac{1-s_t}{s_t} \right)^{\frac{\lambda}{1-\phi}} \\
\implies \left(\frac{z}{y} \right)^{\frac{1}{1-\phi}} \left(\frac{s_t}{1-s_t} \right)^{\frac{\lambda}{1-\phi}} &= \frac{A_t}{B_t} \tag{219}
\end{aligned}$$

Thus:

$$c_t = \frac{\left(\frac{\delta_t}{\delta} \left(\frac{\ell_t}{1-\ell_t} \left(\left(\frac{z}{y} \right)^{\frac{1}{1-\phi}} \left(\frac{s_t}{1-s_t} \right)^{\frac{\lambda}{1-\phi}} \right)^\alpha \right)^{-\beta} \right)^{\frac{1}{\epsilon-\beta}}}{N_t} \tag{220}$$

A.14 Proof of Proposition 13

We know from the FOCs of the Hamiltonian that:

$$\tilde{v}_t = \frac{\tilde{u}_t}{\rho + \delta_t - g_{vt}}, \quad \tilde{u}_t = \frac{u(c_t)}{u'(c_t)c_t} \quad (221)$$

First, consider the denominator of \tilde{v}_t : $\rho + \delta_t - g_{vt}$. As $\delta_t \rightarrow 0$, $\rho + \delta_t - g_{vt} \rightarrow \rho + g_{vt}$.

Next, consider g_{vt} . We know from our FOCs that

$$v_t = \frac{u(c_t)}{\rho + \delta_t - g_{vt}}$$

If the denominator converges to a constant as $\delta_t \rightarrow 0$ and $\tilde{u} \rightarrow \infty$, we get

$$g_{vt} \rightarrow \frac{\dot{u}(c_t)}{u(c_t)} = \frac{u'(c_t)\dot{c}}{u(c_t)} = \frac{u'(c_t)cg_{ct}}{u(c_t)} \quad (222)$$

$$\implies g_{vt} \rightarrow \frac{1}{\tilde{u}_t} g_{ct} \quad (223)$$

This means $g_{vt} \rightarrow 0$ as $\tilde{u} \rightarrow \infty$, so indeed means the denominator v_t converges to a constant.

This implies that as $\delta_t \rightarrow 0$ and $\tilde{u} \rightarrow \infty$:

$$\tilde{v}_t = \frac{\tilde{u}_t}{\rho - \delta_t + g_{vt}} \rightarrow \frac{\tilde{u}_t}{\rho} \quad (224)$$

This immediately implies

$$E_\rho^{\tilde{v}} \rightarrow -1 \quad (225)$$

and

$$E_c^{\tilde{v}} \rightarrow E_c^{\tilde{u}} \quad (226)$$

We now turn to $E_c^{\tilde{u}}$. We know

$$\begin{aligned} \tilde{u}(c_t) &= \bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma} \\ \implies \tilde{u}'(c_t) &= (\gamma-1)\bar{u}c_t^{\gamma-2} \end{aligned} \quad (227)$$

Thus,

$$E_c^{\tilde{u}} = \frac{\tilde{u}'(c_t)c_t}{\tilde{u}(c_t)} = \frac{(\gamma - 1)\bar{u}c_t^{\gamma-1}}{\bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma}} \quad (228)$$

$$\implies E_c^{\tilde{u}} = \frac{(\gamma - 1)\left(\tilde{u}_t + \frac{1}{\gamma-1}\right)}{\tilde{u}} \quad (229)$$

$$\implies E_c^{\tilde{u}} \rightarrow (\gamma - 1) \quad (230)$$

$$\implies E_c^{\tilde{v}} \rightarrow (\gamma - 1) \quad (231)$$

as $\tilde{u}_t \rightarrow \infty$.

Finally, to calculate the ρ' that increases \tilde{v}_t equivalent to a doubling of consumption (starting from ρ^* %) for the table in the main text, I find the ρ' that satisfies:

$$2^{E_c^{\tilde{v}}} = \left(\frac{\rho'}{\rho^*}\right)^{E_c^{\tilde{v}}} \quad (232)$$

$$\implies 2^{(\gamma-1)} = \left(\frac{\rho'}{\rho^*}\right)^{-1} \quad (233)$$

$$\implies \rho' = 2^{(1-\gamma)}\rho^* \quad (234)$$

B Numerical Simulation

B.1 Simulating the Transition Dynamics

I solve the system of differential equations characterizing the optimal allocation numerically using “reverse shooting” (like Jones (2016)). I start from the steady state, consider a small deviation, and then run time backwards. In the notation that follows, I start from time T and run time backwards to time 0.

Given values for the parameters γ , ϵ , β , ρ , λ , ϕ , α , \bar{n} , \bar{u} , and $\bar{\delta}$, as well as a specified N_T and a small $\delta_T > 0$ (small deviation from the steady state), I need to find values of s_T and ℓ_T . To do this, I use the function ‘fminsearch’ in Matlab to find values of s_T and ℓ_T that minimize the distance between \hat{s}_T , $\hat{\ell}_T$ and $\hat{\sigma}_T$ and their steady state values. I then run time backwards, giving us a candidate path.

To determine the values for the other parameters, I first pick ϵ , β , and γ . I also set ϕ . Then, I use the function ‘patternsearch’ in Matlab to find values for λ , $\bar{\delta}$, \bar{u} , N_T and δ_T which minimize the weighted sum of the deviations from a selection of moments of the candidate path and a set of preferred values. These moments are given below.

1. Given a candidate path, I first find the year t_0 in which \tilde{u} , the value of a year of life as a ratio to consumption, is closest to 4. The first moment is \tilde{u}_{t_0} compared to 4.
2. The second moment is the growth rate of consumption at t_0 compared to 1 percent.
3. The third moment is ℓ_{t_0} , the fraction of workers in the consumption sector, compared to 95%.
4. The fourth moment is the growth rate of proportion of the population working as scientists at time t_0 , $g_{\sigma t_0}$, compared to 2%.
5. The fifth moment is hazard rate δ at time t_0 , compared to 0.1%.
6. The sixth moment is the growth rate of δ at time T compared to g_δ , to ensure the the simulation is close to steady state at time T .

I pick $\gamma = 1.5$, $\epsilon = 0.4$, and $\beta = 0.3$ as reasonable parameters. In addition, I set $\rho = 0.02$, $\alpha = 1$, and $\bar{n} = 1\%$. Note that these choices don’t seem to matter for the qualitative results (as long as $\epsilon > \beta$ and $\gamma > 1$). The process described above can find different local minima depending on the initial guess as well the value of ϕ supplied, so I hunt for the best overall fit. I end up using $\phi = 5/6$ and $\lambda = 0.3$, $\bar{\delta} = 3.8965 \times 10^{-5}$, $\bar{u} = 0.0098$,

$N_T = 9.2955 \times 10^{14}$, and $\delta_T = 5 \times 10^{-4}$.

To extrapolate M_∞ , I have to calculate the area under the hazard rate curve, i.e. $\int_{t_0}^{\infty} \delta_s ds$. Note that:

$$\begin{aligned} \int_{t_0}^{\infty} \delta_s ds &= \int_{t_0}^T \delta_s ds + \int_T^{\infty} \delta_s ds \\ \Rightarrow \int_{t_0}^{\infty} \delta_s ds &\approx \int_{t_0}^T \delta_s ds + \int_0^{\infty} \delta_T \cdot e^{-sg\delta} ds \\ \Rightarrow \int_{t_0}^{\infty} \delta_s ds &\approx \int_{t_0}^T \delta_s ds + \frac{\delta_T}{g\delta} \end{aligned} \quad (235)$$

since at time T , we are approximately at the steady state, where δ declines exponentially.

Thus, I sum the area under our simulated δ from the time representing today to the end of the simulation, which gives us $\int_{t_0}^T \delta_s ds$. Verifying that indeed $\hat{\delta}_T \approx g\delta$, I can then calculate $\frac{\delta_T}{g\delta}$. Summing these two terms gives us the desired $\int_{t_0}^{\infty} \delta_s ds$, and then the probability of humanity surviving to infinity conditional on surviving until t_0 is then $e^{-\int_{t_0}^{\infty} \delta_s ds}$.

B.2 Simulating the Acceleration in Growth

The natural way to simulate the acceleration of growth (in this case, faster population growth) would be to solve the differential equations characterizing the optimal allocation using “forward shooting”. However, due to the instability of the system of differential equations, this yields unreliable results. Thus, I again proceed by solving the system of differential equations using “reverse shooting” as when we simulated the transition dynamics.

First, the transition path without the acceleration in growth is given by the path as found in Appendix B.1. I will refer to this as the unperturbed path.

Next, we would like to simulate the transition path with accelerated growth. I use the same parameters as in Appendix B.1 except for a time-varying rate of \bar{n} , set as discussed in the main text. This gives me a candidate path with accelerated growth. I would like to find a transition path with acceleration that matches the unperturbed path up until the moment of acceleration. Thus, using the function ‘fminsearch’ in Matlab, I find δ_T and N_T that yield a candidate path with accelerated growth that minimizes

the weighted sum of the deviations from a selection of moments and a set of preferred values.

In particular, I pick some year t_0 prior to the year in which growth accelerates; this will be the reference year on the unperturbed path. Given a candidate path, I find a year t^* in which δ_{t^*} of our candidate path is closest to δ_{t_0} on the unperturbed path. Then, my moments are s_{t^*} , ℓ_{t^*} , σ_{t^*} , δ_{t^*} , y_{t^*} , z_{t^*} , and N_{t^*} , compared to their respective values at t_0 on the unperturbed transition path. Since s , ℓ , σ , δ , y , z , and N uniquely characterize all the variables of our economy on the optimal allocation and both the unperturbed and the accelerated path evolve according to the same system of differential equations prior to the acceleration, this ensures that both the unperturbed and accelerated transition path represent the same economy up until the moment where growth is accelerated.

I experiment with the weights and the reference year to hunt for the best overall fit. I end up picking $\delta_T = 5.0326 \times 10^{-4}$ and $N_T = 9.3991 \times 10^{14}$ for the accelerated transition path that results in a permanent level effect, and $\delta_T = 5.0001 \times 10^{-4}$ and $N_T = 9.2948 \times 10^{14}$ for the transition path with the temporary boom.

This method appears to work well (i.e. matches the unperturbed path very well) for an acceleration in growth that is not too large, although it is still imperfect. However, it enables us to sidestep the difficulty of “forward shooting” and compare the transition paths with and without an acceleration in growth using “reverse shooting”.

I extrapolate the long-term survival probability M as before.

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