Quadratic Funding with Incomplete Information

Luis V. M. Freitas (Global Priorities Institute, University of Oxford), Wilfredo L. Maldonado (University of Sao Paulo)

Global Priorities Institute | August 2022

GPI Working Paper No . 10-2022

Quadratic Funding with Incomplete Information

Luis V. M. Freitas *[1] and Wilfredo L. Maldonado[2]

[1]Global Priorities Institute, University of Oxford

[2]University of São Paulo

August 11, 2022

Abstract

Quadratic funding is a public good provision mechanism that satisfies desirable theoretical properties, such as efficiency under complete information, and has been gaining popularity in practical applications. We evaluate this mechanism in a setting of incomplete information regarding individual preferences, and show that this result only holds under knife-edge conditions. We also estimate the inefficiency of the mechanism in a variety of settings and show, in particular, that inefficiency increases in population size and in the variance of expected contribution to the public good. We show how these findings can be used to estimate the mechanism's inefficiency in a wide range of situations under incomplete information.

Keywords: Public goods provision, incomplete information, quadratic funding mechanism.

JEL Classification Codes: C72, D82, H41.

Acknowledgements

Wilfredo L. Maldonado would like to thank the financial support of the Fundação Instituto de Pesquisas Econômicas - FIPE and the CNPq of Brazil 306473/2018-6. We thank Gustav Alexandrie, Maya Eden, Loren Fryxell, Zoë Hitzig, Rossa O'Keeffe-O'Donovan, Lennart Stern, Benjamin Tereick and seminar participants at the Global Priorities Institute for the valuable comments on an earlier draft of this paper. All remaining errors are our own.

*Corresponding author. Global Priorities Institute, Universtiy of Oxford, Trajan House, Mill Street, Oxford, OX2 0BS, United Kingdom. Email: luis.motafreitas@economics.ox.ac.uk

1 Introduction

The non-excludability and non-rivalry of public goods poses a challenge for public good provision that has long received considerable attention in both the theoretical and the applied economic literature (Samuelson 1954; Lindahl 1958). Several mechanisms for providing efficient levels of a public good have been proposed (Clarke 1971; Groves and Ledyard 1977; Hylland and Zeckhauser 1979; Walker 1981), and while these mechanisms are of considerable theoretical importance, there has been to date limited practical application of these solutions, resulting at least in part from undesirable properties they were shown to possess (Walker 1981; Healy 2006; Rothkopf 2007). On the other hand, there are various solutions commonly used in practice, such as majority voting, 1:1 donation matching, and private provision of public goods, all of which lead to inefficient outcomes in the general case (Bergstrom 1981; Bergstrom et al. 1986).

The quadratic funding (QF) mechanism, proposed by Buterin et al. (2019), appears to be promising in both a theoretical and a practical sense. This mechanism provides a public good level that is equal to the square of the sum of the square roots of individual contributions. That is, if every individual i contributes some quantity $c_i \geq 0$ for funding a public good, then the resulting funding for the public good through QF is $\left(\sum_i \sqrt{c_i}\right)^2$. Besides efficiency under complete information, one characteristic that distinguishes this mechanism from previously proposed ones is that it does not require any assumptions about the set of public goods to be funded, making it particularly well-suited to cases in which it is important that individuals be able to propose new public goods. It also stands out for its simplicity, and satisfies other desirable properties such as individual rationality and homogeneity of degree one. These characteristics make QF particularly promising for usage in a broad range of situations. In fact, QF has been employed to allocate significant sums of money for funding open-source software projects and matching donations to charity.[1]

This paper aims to analyze the efficiency of the quadratic funding mechanism in a more general informational context, and we do so in two main ways. First, we adapt the framework introduced by Buterin et al. (2019) to allow for the possibility of incomplete information regarding individual preferences. Besides showing the existence of equilibria, we present necessary and sufficient conditions for efficiency, and show that QF is only efficient under knife-edge conditions, which stands in

[1] Gitcoin (https://gitcoin.co/fund) and HackerLink (https://hackerlink.io/en) are two of the main platforms that currently use QF for funding software development, and WeTrust (https://blog.wetrust.io/conclusion-of-the-first-lr-experiment-709b018b5f83) has used QF for matching donations.

contrast with the efficiency of the mechanism under complete information. In particular, we show that QF is inefficient whenever an individual is uncertain about whether the efficient provision is positive, and that QF is efficient for individuals with isoelastic utility functions for the public good if and only if the elasticity coefficient of these functions is equal to 1/2. We show that the latter condition can be interpreted as saying that QF is efficient when the optimal individual contributions is a dominant strategy, i.e., do not depend on the contribution by others, thus presenting an easily verifiable test for efficiency in applications of this mechanism.

Second, and motivated by the large class of models in which the private provision of the public good is inefficient, we use numerical estimations to quantify the inefficiency of QF under incomplete information. We define two measures of inefficiency, and then analyze how these measures respond to changes to parameters of our setup. We show that inefficiency is increasing in the number of players and in the variance of the expected value of the fund, and we characterize conditions under which this response is more or less intense. The results presented in our analysis can be used to assess how QF would perform even when it does not lead to efficient public good provision.

Besides the importance of these findings to quadratic funding, our results also bring implications to quadratic voting (Lalley and Weyl 2019), and more broadly to the growing literature on quadratic pricing (Tideman and Plassmann 2017). Quadratic voting is a voting mechanism that has gained attention both from academia (Kaplow and Kominers 2017; Park and Rivest 2017; Quarfoot et al. 2017; Weyl 2017) and from policymakers, having been applied by the Democratic Party of the United States for political decision-making.[2] Quadratic funding can be understood as an adaptation of quadratic voting to a context of continuous public good provision, which makes our findings particularly surprising, given that Lalley and Weyl (2019) showed that the outcome chosen through quadratic voting under incomplete information converges to efficiency as the population grows. Therefore, our results suggest that incomplete information might pose an important challenge to other quadratic pricing mechanisms, despite the efficiency result for quadratic voting, and, more generally, the properties pertaining to one mechanism may fail to be held by the other.

The paper is divided into seven sections. Section 2 presents the setting used in the paper, and shows the existence of equilibria for quadratic funding. Section 3 presents efficiency results under complete information, and section 4 analyzes efficiency under incomplete information. Section 5 gives an economic intuition and an efficiency result for the special case where individuals have isoelastic utility functions for the public good, and section 6 employs this class of utility function to

[2] https://www.wired.com/story/colorado-quadratic-voting-experiment/

develop quantitative estimates of inefficiency under incomplete information. Section 7 summarizes some conclusions of the paper. The Appendix provides the proofs of the stated propositions.

2 Setting

There exist $I \geq 2$ individuals identified by the elements in the set $\mathcal{I} = \{1, 2, \cdots, I\}$ and a single public good supplied in continuous amounts and denoted by $F \in \mathbb{R}_+$. For each $i \in \mathcal{I}$ there exists a finite set of types $\Theta_i = \{\theta_i^1, \cdots, \theta_i^{L_i}\}$ associated, and an expected utility function $u_i : \mathbb{R}_+ \times \mathbb{R} \times \Theta_i \to \mathbb{R}$ defined as $u_i(F, m; \theta_i) = v_i(F; \theta_i) + m$, where the linear good is the numeraire and $v_i : \mathbb{R}_+ \times \Theta_i \to \mathbb{R}$ represents the monetary-equivalent expected utility of i for a level $F \geq 0$ of funding for the public good when her type is $\theta_i \in \Theta_i$. We use $\Theta = \times_{i=1}^{I} \Theta_i$ to denote the set of type profiles. The joint probability distribution of types is $\Pr : \Theta \to [0, 1]$ and is assumed to be common knowledge. We make use of the following assumptions, where Assumption 1 imposes some standard requirements on the utility functions, and Assumption 2 guarantess that every type profile occurs with positive probability.

Assumption 1. For every $i \in \mathcal{I}$ and $\theta_i \in \Theta_i$, $v_i(\cdot; \theta_i) \in C^1$ is a strictly increasing and strictly concave function, and $\lim_{F \to \infty} v_i'(F; \theta_i) = 0$.

Assumption 2. The image of the joint probability distribution of types Pr is strictly positive.

We now proceed to state the definitions used throughout this paper.

Definition 1. A *funding mechanism* is a function $\Phi : \mathbb{R}_+^I \to \mathbb{R}_+$ that determines, for any contribution profile of the individuals to the public good $\mathbf{c} := (c_1, c_2, \cdots, c_I) \in \mathbb{R}_+^I$, a level of public good provision $\Phi(\mathbf{c}) = F \in \mathbb{R}_+$.[3]

In words, a funding mechanism is a technology that maps individual contributions (inputs) to a level of the public good to be provided (output). The simple and classical funding mechanism is the linear technology $\Phi(\mathbf{c}) := \sum_{i=1}^{I} c_i$. Note that it need not be the case that the mechanism is budget balanced (i.e., that $\Phi(\mathbf{c}) = \sum_{i=1}^{I} c_i$ holds), and so a funding mechanism might require external funding or generate a budget surplus.

[3] In more general terms, a funding mechanism could be defined a function $y : \times_{i=1}^{I} \mathcal{M}_i \to X \times \mathbb{R}_+^I$ such that for all i, $\mathcal{M}_i = X = \mathbb{R}_+$, defined by $y(\mathbf{c}) = (\Phi(\mathbf{c}), \mathbf{c})$ with Φ defined as in definition 1. We refer to Φ as the funding mechanism to simplify the notation.

The technology defining the funding mechanism Φ may be a general one with some suitable properties. Homogeneity of degree one ($\Phi(\lambda\mathbf{c}) = \lambda\Phi(\mathbf{c})$) guarantees the irrelevance of the units scale used to measure the contributions. Anonymity ($\Phi(\mathbf{c}) = \Phi(\sigma(\mathbf{c}))$, where $\sigma(\cdot)$ is a permutation operator) guarantees the irrelevance of the order of contributors. Inada's condition for individual i when someone else is contributing with a strictly positive amount ($\lim_{c_i \to 0} \Phi_i(c_i, \mathbf{c}_{-i}) = +\infty$, where $\mathbf{c}_{-i} \neq \mathbf{0}$ and Φ_i is the partial derivative of Φ with respect to the ith component) incentivizes individual i to contribute whenever other individuals are providing positive contributions to the public good. A technology with all these properties is the CES technology $\Phi(\mathbf{c}) = \left[\sum_{i=1}^{I} c_i^{\rho}\right]^{1/\rho}$, with $\rho < 1$ to guarantee the strict convexity of the technology. To this family of technologies belongs the QF mechanism.

Definition 2. The *quadratic funding mechanism* is defined by the function $\Phi^{QF} : \mathbb{R}_+^I \to \mathbb{R}_+$ given by:

$$\Phi^{QF}(\mathbf{c}) = \left(\sum_{i=1}^{I} (c_i)^{1/2}\right)^2 .$$

Whenever possible, a central planner would choose a funding mechanism that fosters individual contributions generating efficient output levels. However, before delving into efficiency, let us define the concept of a public good contribution game with a funding mechanism and its corresponding Nash equilibrium.

Definition 3. A *public good provision game* with funding mechanism Φ is defined by

$$\mathcal{G} = \left\{ (v_i, \Theta_i)_{i \in \mathcal{I}}, \Pr, \Phi \right\}.$$

The equilibrium concept for public good provision games are defined as follows.

Definition 4. A profile $\mathbf{c}^* = (c_1^*, \cdots, c_I^*)$, where for all $i \in \mathcal{I}$, $c_i^* : \Theta_i \to \mathbb{R}_+$ is an *equilibrium* for $\mathcal{G}$ if, for each $i \in \mathcal{I}$ and each $\theta_i \in \Theta_i$, the following is satisfied for all $z \in \mathbb{R}_+$:

$$E\left[v_i(\Phi(c_i^*(\theta_i), \mathbf{c}_{-i}^*(\theta_{-i})); \theta_i) \,\middle|\, \theta_i\right] - c_i^*(\theta_i) \geq E\left[v_i(\Phi(z, \mathbf{c}_{-i}^*(\theta_{-i})); \theta_i) \,\middle|\, \theta_i\right] - z.$$

Alternatively, for all $i \in \mathcal{I}$ and all $\theta_i \in \Theta_i$, $\mathbf{c}^*$ satisfies:

$$c_i^*(\theta_i) \in \arg\max_{z \in \mathbb{R}_+} E\left[v_i(\Phi(z, \mathbf{c}_{-i}^*(\theta_{-i})); \theta_i) \,\middle|\, \theta_i\right] - z.$$

An equilibrium $\mathbf{c}^*$ is called *interior* if $c_i^*(\cdot) > 0$, for all $i \in \mathcal{I}$.

Now, let us turn to the definition of efficiency. The social welfare function in this context is the function assigning to each level of public good provision its social net value given a type vector, namely,

$$W(F;\theta) = \left(\sum_{i=1}^{I} v_i(F;\theta_i)\right) - F.$$

From a normative standpoint, a desirable property for a funding mechanism is the ability to generate private contributions that attain efficient levels of the public good for any given vector of types. We formalize these notions in the following definitions.

Definition 5. We say that $F^e : \Theta \to \mathbb{R}_+$ is an *(ex-post) efficient provision* for $\mathcal{G}$ if, for all $\theta \in \Theta$,

$$F^e(\theta) \in \arg\max_{F \geq 0} \left(\sum_{i=1}^{I} v_i\left(F;\theta_i\right)\right) - F.$$

Definition 6. The funding mechanism Φ is *efficient* for $\mathcal{G} = \left\{(v_i, \Theta_i)_{i\in\mathcal{I}}, \Pr, \Phi\right\}$ if there exists an equilibrium contribution profile $\mathbf{c}^*$ such that, for all $\theta \in \Theta$, $\Phi(\mathbf{c}^*(\theta)) = F^e(\theta)$ is an efficient provision for $\mathcal{G}$.

We conclude this section by showing that Assumption 1 is sufficient to guarantee the existence of an equilibrium, a result which will be useful in our analysis of efficiency of equilibria in the following sections.

Proposition 2.1. *Suppose that assumption 1 holds. Then, there exists a unique efficient provision* $F^e : \Theta \to \mathbb{R}_+$.

Proposition 2.2. *Suppose that assumption 1 holds. Then, there exists an equilibrium for the quadratic funding mechanism.*

3 Efficiency under complete information

In this paper, we use the term *complete information* to refer to games where the type set is a singleton, as stated below.

Definition 7. A game is said to have *complete information* if $|\Theta| = 1$.

The setting and definitions presented in the previous sections are very similar to those used by Buterin et al. (2019) when considering games of complete information. As we are building on their work on quadratic funding, we summarize their main result in the proposition below.

Proposition 3.1 (Buterin et al. 2019, adapted). *Let $\mathcal{G}$ be a game of complete information where, for all $i \in \mathcal{I}$, $v_i \in C^1$ is a strictly increasing and strictly concave function, and the funding mechanism is a CES mechanism $\Phi(\mathbf{c}) = \left[\sum_{i=1}^{I} c_i^{\rho}\right]^{1/\rho}$ with $\rho < 1$. If $\mathbf{c} \gg 0$ is an interior equilibrium of this mechanism and the efficient allocation is $F^e > 0$, then:*

(i) If $\rho > 1/2$, then $\Phi(\mathbf{c}) < F^e$;

(ii) If $\rho < 1/2$, then $\Phi(\mathbf{c}) > F^e$;

(iii) If $\rho = 1/2$, then $\Phi(\mathbf{c}) = F^e$.

Thus, among the CES funding mechanisms, only QF can generate efficient levels of the public good for interior solutions. For this reason, we will center our analysis on QF throughout the remainder of this paper.

Proposition 3.1 is not sufficient to guarantee efficiency in the sense introduced by definition 6. This is because, on the one hand, it only considers interior equilibria, and so it is not applicable to the case where one or more individuals do not contribute to the public good. On the other hand, this proposition assumes the existence of an efficient allocation and an equilibrium allocation without presenting conditions under which those allocations would exist.

If, in addition to the hypotheses made in proposition 3.1 about the utility function, we assume that $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$, we get Assumption 1, and can thus apply proposition 2.1. With this additional assumption, it is possible to show that QF is efficient.

Proposition 3.2. *In a game of complete information, suppose that Assumption 1 holds. Then, the quadratic funding mechanism is efficient.*

4 Efficiency under incomplete information

In this section, we consider games that do not necessarily have complete information, that is, we allow for $|\Theta| > 1$. We show that, unlike efficiency under complete information, when individuals are uncertain of the preferences of their peers, efficiency only results under strong conditions on the utility functions. In what follows, we will establish a couple of necessary and sufficient conditions for this property to hold.

Proposition 4.1. *Suppose that assumptions 1 and 2 hold, and that there exists some $\theta' \in \Theta$ for which the efficient public good provision level is zero. Then, the quadratic funding mechanism is efficient if and only if the efficient public good provision level is zero for all $\theta \in \Theta$.*

Proposition 4.1 shows that there exists a broad range of public good games with quadratic funding where the equilibrium is not efficient. It suffices to exist two type profiles, one with zero efficient provision and another with positive efficient provision, to have inefficiency of the equilibrium. Thus, to look for the cases where QF is efficient, let us consider games where the efficient level of the public good is strictly positive and try to find conditions on the preferences that allows for the efficiency of the equilibrium. The following proposition states those conditions when only one individual has multiple types.

PROPOSITION 4.2. *Suppose that assumptions 1 and 2 hold, that efficient public good provision level is strictly positive for all $\theta \in \Theta$, and that there is only one individual with more than one possible type; namely, there exists a unique $j \in \mathcal{I}$ such that $|\Theta_j| > 1$. Then, the quadratic funding mechanism is efficient if, and only if, there exists $A \in \mathbb{R}$ such that $\sum_{i \neq j} v_i'(F^e(\theta); \theta_i) = A(F^e(\theta))^{-1/2}$ for all $\theta \in \Theta$. Furthermore, we have that $A = \sum_{i \neq j}(c_i^*(\theta_i))^{1/2}$.*

Figure 1 illustrates this proposition. Starting from the sum of marginal utilities of the individuals with a unitary type set and choosing a type vector $\theta \in \Theta$, we can choose an $A \in \mathbb{R}$ for which $\sum_{i \neq j} v_i'(F^e(\theta); \theta_i) = A(F^e(\theta))^{-1/2}$ holds. Using this A, we can plot the function $A(F)^{-1/2}$. Proposition 4.2 implies that, for efficiency to hold, it must be the case that $A(F)^{-1/2}$ and $\sum_{i \neq j} v_i'(F; \theta_i)$ intersect whenever $F = F^e(\theta')$ for some $\theta' \in \Theta$.

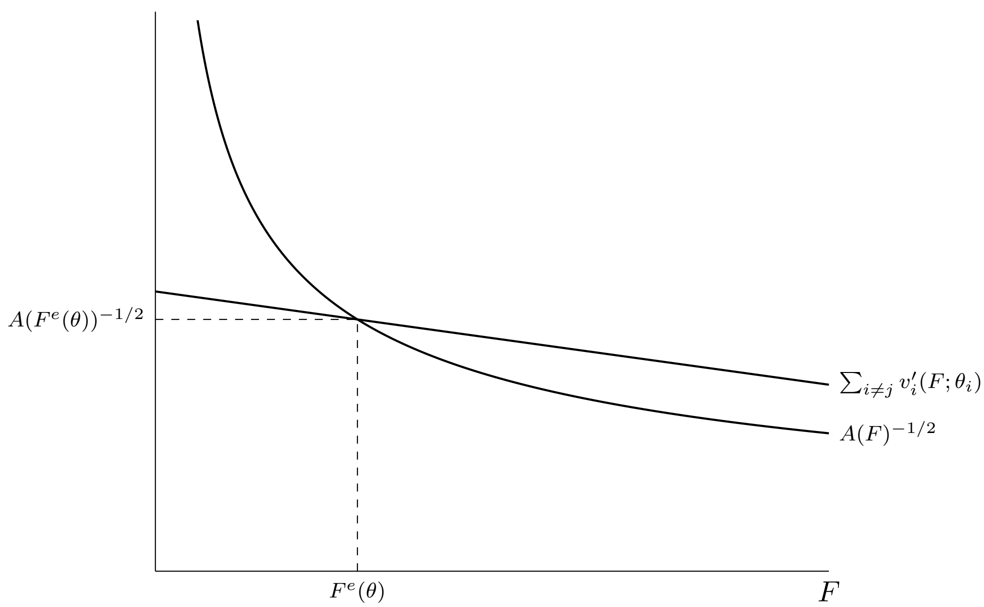

Figure 1: Illustration of proposition 4.2.

As this figure indicates, this necessary and sufficient condition for efficiency is a knife-edge condition, and will generally not be true for utility functions satisfying assumption 1. Furthermore, this proposition assumes what is, in fact, the simplest case where individuals have private information, where only a single individual is allowed to have multiple types. Thus, this proposition shows that knife-edge conditions apply even in this very simple case, implying that cases where multiple individuals have non-unitary type sets have conditions for efficiency that are at least as strict as this one.

A natural question arising from this result is whether efficiency can hold at all in cases where multiple individuals have non-unitary type sets, and the answer turns out to be positive. It is easy to see that the condition in proposition 4.2 implies that, in this case, efficiency follows if every individual with a single type has a utility function for the public good that is equal to an isoelastic utility function with an elasticity parameter $\eta = 1/2$, as their marginal utility function for the public good will be of the form $A(F)^{-1/2}$ for some $A \in \mathbb{R}$. In the next section, we analyze the special case of isoelastic utility functions, which will help us better understand this efficiency condition and show that efficiency under incomplete information always holds when every individual has an isoelastic utility function with $\eta = 1/2$.

5 Isoelastic utility functions

In this section, we show that isoelastic utility functions for the public good represent a special class of utility functions for which the individual's best-response contribution function is increasing or decreasing in the contributions of others depending entirely on the value of the elasticity parameter η. We also show that, when every individual has an isoelastic utility function for the public good with $\eta = 1/2$ for any type vector, efficiency under incomplete information holds.

First, consider a situation with complete information. Let $i = 1$ index an individual with utility function for the public good $v_1(F) = \beta F^{1-\eta}/(1-\eta)$, if $\eta \neq 1$, and $v_1(F) = \beta \ln(F)$, if $\eta = 1$. The first order condition defining c_1, the best-response of this individual to the contribution profile $\mathbf{c}_{-1}$ of all the other individuals, is:

$$\frac{c_1^{1/2}}{\sum_{j=1}^{I} c_j^{1/2}} = \frac{\beta}{\left(\sum_{j=1}^{I} c_j^{1/2}\right)^{2\eta}}. \tag{1}$$

The right-hand-side of equation (1) represents the marginal utility of increasing the provision of the public good by one (infinitesimal) unit, whereas the left-hand side is the marginal increase in the individual contribution per unit of increase in the level of the public good.

To analyze the best-response variation to changes in the contributions of other individuals, let us rewrite equation (1) as follows:

$$c_1^{1/2} = \beta \left(c_1^{1/2} + \sum_{j \neq 1}^{I} c_j^{1/2} \right)^{1-2\eta}.$$

Thus, if $\eta > 1/2$, then an increase in the aggregate square-roots of the others' contributions will produce a reduction in the best-response c_1. The reciprocal effect holds if $\eta < 1/2$. Finally, if $\eta = 1/2$, the best-response of individual $i = 1$ does not depend on the contributions of her peers.

To summarize the above analysis, we state the following proposition.

Proposition 5.1. *Suppose that an individual has an isoelastic utility function for the public good with elasticity coefficient $\eta > 0$. Then, that individual's contribution to the public good is a decreasing (increasing) function of the aggregate square-roots of the other individuals' contributions if, and only if, $\eta \geq 1/2$ ($\eta \leq 1/2$).*

A conclusion that results from proposition 5.1 is the stabilizer response of an individual with isoelastic utility for the public good to variations in the contributions of her peers. If her elasticity coefficient is $\eta > 1/2$, her best response function is submodular, and so increases in the contributions of others lead her to contribute less to the public good. On the other hand, with $\eta < 1/2$, her best response function is supermodular, and increases in the contributions of others lead her to contribute more to the public good. In the limit case ($\eta = 1/2$), she is indifferent to variations of the contributions of others.

Now turning to the case of incomplete information, we state the following result.

Proposition 5.2. *Suppose that assumption 2 holds, and that for every $i \in \mathcal{I}$ and $\theta_i \in \Theta_i$, we have that $v_i(F; \theta_i) = \beta_i(\theta_i) F^{1-\eta}/(1-\eta)$, where $\beta_i(\theta_i) > 0$ and $\eta > 0$. Additionally, suppose that for some $j \in \mathcal{I}$, there exist $\theta_j^k, \theta_j^\ell \in \Theta_j$ such that $\beta_j(\theta_j^k) \neq \beta_j(\theta_j^\ell)$. Then, the quadratic funding mechanism is efficient if, and only if, $\eta = 1/2$.*

In words, this proposition says that, when every individual of every type has an isoelastic utility function for the public good with equal η, and at least one individual has two distinct utility functions, then efficiency holds if, and only if, $\eta = 1/2$. By proposition 5.1, when an individual has an isoelastic utility function for the public good with $\eta = 1/2$, their best response does not depend on the contribution of others. When this holds for every individual and every type, then every individual always chooses exactly the same contribution as they would choose under complete information. As QF is efficient under complete information, it follows that efficiency ex-post will hold.

On the other hand, if every individual has the same $\eta \neq 1/2$, then their optimal response depends on the contribution chosen by other individuals. The proposition assumes that at least one individual has two distinct utility functions, implying that some individuals face incomplete information about how much one individual will contribute, and so from assumption 2 it follows that some individuals face incomplete information about the aggregate contribution of others. Thus, in different states of the world, an individual facing incomplete information would want to choose different levels of contribution if he were to know the true state of the world. As the utility function of every individual has the same elasticity coefficient, if $\eta < 1/2$ ($\eta > 1/2$), then in the type vector where every individual values the public good the most, by proposition 5.1 the public good will be underfunded (overfunded), and so efficiency ex post will not follow.

In the next section, we turn to quantitative estimates of the inefficiency of quadratic funding, making use of isoelastic utility functions and building on the intuition developed here.

6 Inefficiency estimates

Due to the generic inefficiency of the equilibrium that QF produces when the game has incomplete information, in this section we analyze the size of such inefficiency as well as its sensitivity to variations in fundamental parameters of the model. We are going to define two measures for the size of inefficiency, and analyze their behavior when either population size increases or underlying uncertainty changes.

Before defining the measures of inefficiency, let us introduce the notion of "*second-best*" provision level of the public good in a game $\mathcal{G} = \{(v_i)_{i \in \mathcal{I}}, \Phi\}$.

Definition 8. We say that $F^{EA} \geq 0$ is an *ex-ante efficient provision* for $\mathcal{G}$ if

$$F^{EA} = \underset{F \geq 0}{\arg\max}\, E\left[\left(\sum_{i=1}^{I} v_i\left(F; \theta_i\right)\right) - F\right].$$

That is, if it maximizes the expected social welfare before the individual types are known.

Notice the difference between the ex-ante and the ex-post efficient provision given in definition 5. The ex-ante efficient provision represents the efficient level of public good provision when the probability distribution of types is used to measure the welfare of the society. Consequently, this provision level is the best that the central planner could choose without using any mechanism to acquire information about the types.

Having defined both ex-ante and ex-post efficient provision levels of the public good, we can define our measures for the size of inefficiency.

Definition 9. The *absolute deadweight loss* for the contribution-based quadratic funding mechanism in the game $\mathcal{G}$ is

$$\Delta W^A := E\left[W(F^e(\theta)) - W(\Phi^{QF}(\mathbf{c}^*(\theta)))\right].$$

In words, the absolute deadweight loss is the expected loss in monetary terms of using the QF equilibrium to fund the public good instead of using the efficient level of the public good for each profile of types that individuals may have.

Definition 10. The *relative deadweight loss* for the contribution-based quadratic funding mechanism in the game $\mathcal{G}$ is

$$\Delta W^R := \frac{E\left[W(F^e(\theta)) - W(\Phi^{QF}(\mathbf{c}^*(\theta)))\right]}{E\left[W(F^e(\theta)) - W(F^{EA})\right]}.$$

The relative deadweight loss measures the ratio between the absolute deadweight loss and the deadweight loss of providing the ex-ante efficient level of the public good rather than the ex-post efficient level. The intuition behind this measure is that, by employing the quadratic funding mechanism, the social planner makes use private information about the individual types, whereas by choosing the ex-ante efficient provision, no private information about individual types is gained. One would, therefore, expect QF to be comparatively more efficient than this benchmark provision, in expectation, in situations where it is to be employed. This is captured by a measure of $\Delta W^R < 1$, whereas $\Delta W^R > 1$ denotes situations where QF is more inefficient than the ex-ante optimal provision.

Measures similar to that presented in definition 9 are commonly used in the literature (e.g., see Vives 2002; Rustichini et al. 1994). However, to the best of our knowledge, this is the first time that a measure like the one in definition 10 is proposed.

In the next two subsections, we will analyze the response of the inefficiency measures proposed above to changes in the number of participants in the society and in the level of uncertainty contained in the incomplete information of the game.

6.1 Changes in the population size

To analyze the welfare changes as a response to population size increases, we consider an example where each individual may have one of two types with the same utility functions. Specifically, for $i \in \mathcal{I}$, let $|\Theta_i| = 2$, let $\Pr(\theta_i = \theta_i^1) = \Pr(\theta_i = \theta_i^1 \,|\, \theta_j) = 1/2$ for all $i, j \in \mathcal{I}$ and $i \neq j$, and let their utility functions of consuming the public good for each type be given by

$$v_i(F; \theta_i^1) = \frac{F^{1-\eta}}{1-\eta},$$

$$v_i(F; \theta_i^2) = 2\frac{F^{1-\eta}}{1-\eta}.$$

Given the above setup, we have that the probability of $0 \leq k \leq I$ individuals being of type 1 follows a binomial distribution. As individuals are symmetric, their contributions to the public good are identical whenever they have the same type, so let us use the notation $x^1 := c_i(\theta_i^1)$ and $x^2 := c_i(\theta_i^2)$ to refer to these contributions. The first order conditions for the problem of an individual with type 1 can be written as

$$(x^1)^{1/2} = \frac{1}{2^{I-1}} \sum_{k=0}^{I-1} \binom{I-1}{k} \left[(k+1)(x^1)^{1/2} + (I-1-k)(x^2)^{1/2}\right]^{1-2\eta}. \quad (2)$$

Respectively, for an individual with type 2:

$$(x^2)^{1/2} = \frac{1}{2^{I-1}} \sum_{k=0}^{I-1} \binom{I-1}{k} 2\left[(k(x^1)^{1/2} + (I-k)(x^2)^{1/2}\right]^{1-2\eta}. \quad (3)$$

Solving equations (2) and (3) numerically, we obtain the QF equilibrium for this game.

The ex-post efficient provision level depends solely on the number of individuals with a certain type. For a type profile $\theta \in \Theta$ where the number of individuals with type 1 is $0 \leq k \leq I$, the efficient provision solves

$$\max_{F \geq 0} k\frac{F^{1-\eta}}{1-\eta} + 2(I-k)\frac{F^{1-\eta}}{1-\eta} - F,$$

whose first order conditions imply that

$$F^e(\theta) = (2I-k)^{1/\eta}. \quad (4)$$

And finally, the ex-ante efficient provision level must solve

$$\max_{F \geq 0} \left[\frac{1}{2^I} \sum_{k=0}^{I} \binom{I}{k} (k + 2(I-k))\frac{F^{1-\eta}}{1-\eta}\right] - F,$$

whose explicit solution is

$$F^{EA} = \left[\frac{1}{2^I} \sum_{k=0}^{I} \binom{I}{k} (2I - k) \right]^{1/\eta}. \tag{5}$$

With the analytic forms found above, we are going to check if the inefficiency converges to zero as the population size increases, a condition frequently evaluated in the literature (Lalley and Weyl 2019; Rustichini et al. 1994). We also discuss the asymptotic behavior of the relative deadweight loss as the number of participants goes to infinity.

First, let us think about the two evident effects that an increase in the population size brings to the equilibrium. One of them is that a larger population increases, *ceteris paribus*, the number of contributions to the public good, which in turn would augment the variance of the provision level. This effect exacerbates the problem generated by incomplete information, resulting in an increase in deadweight loss. On the other hand, a second effect takes place, which is dependent on η. When I increases, as the extra individuals make positive contributions, this would *ceteris paribus* raise the level of funding for the public good. However, proposition 5.1 asserted (in the complete information case) that different values of η result in different responses of individual contributions to changes in aggregate provision. Namely, if $\eta > 1/2$, individuals reduce their contributions, converging to zero as I approaches infinity. This promotes a reduction in the dispersion of contributions, and thus lowers the problem caused by incomplete information. The opposite occurs when $\eta < 1/2$, so the variance of contribution increases, contributing to an increase in inefficiency. From now on, we will refer to these effects as *contributor quantity* effect and *contribution dispersion* effect, respectively.

Figure 2 presents our plots for the absolute deadweight loss as a function of the population size. In an attempt to isolate the contributor quantity effect, we start by analyzing the deadweight loss for the case where $\eta = 0.499$. The reason for doing so is that this value is close enough to $\eta = 1/2$ for the contribution dispersion effect to be comparatively small, while also being different from $\eta = 1/2$ and thus having a non-zero deadweight loss for $I \geq 2$. As we can see in, the deadweight loss in this case is approximately linear.

For $\eta = 1/4$, and in line with our previous analysis, the contribution dispersion effect is such that it contributes to a higher inefficiency as population grows. The combination of a positive contributor quantity effect and a positive contribution dispersion effect makes the total absolute deadweight loss grow at increasing rates, reaching by far the largest order of magnitude of all plots in this figure.

On the other hand, when $\eta = 1$, the contribution dispersion effect lowers the rate of growth of

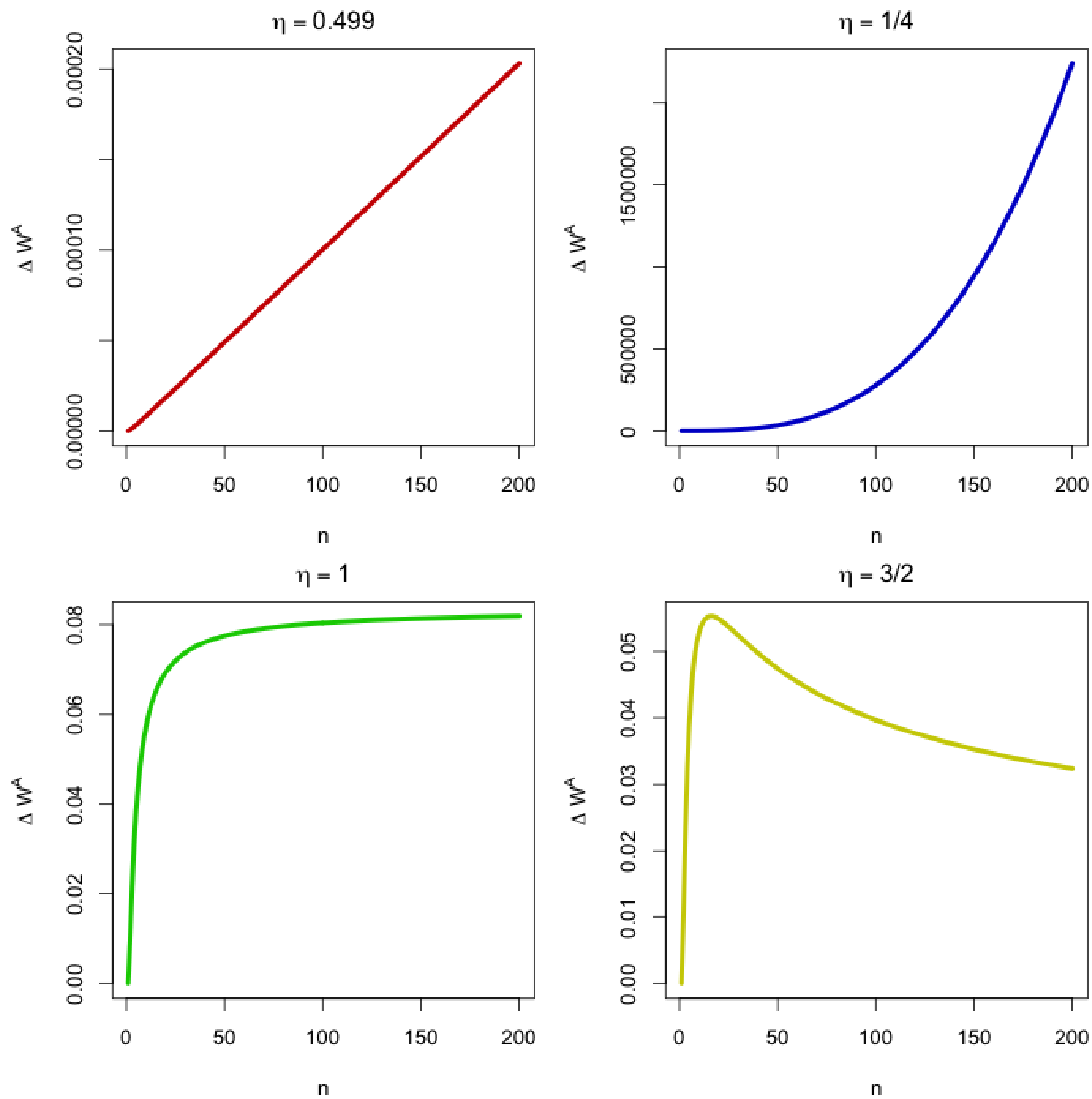

Figure 2: Absolute deadweight loss as a function of $1 \leq I \leq 200$.

inefficiency. The intensity of the effect is large enough to make the deadweight loss converge to a positive value, but its magnitude is not sufficient to counter the contributor quantity effect, and hence QF is not asymptotically efficient. However, when we set $\eta = 1.5 > 1$, the intensity of the contribution dispersion effect becomes strong enough to cause QF to become more efficient as the population size grows, possibly converging to zero.

We now show the relative deadweight loss in figure 3. Notice that all these graphs exhibit the same shape, growing at decreasing rates and (apparently) converging to a positive value. This highlights the advantage of QF in letting individuals choose their contributions using their private information; however, a larger population diminishes the relative importance of the information possessed by each individual, thus lowering the comparative efficiency of this mechanism.

Let us comment each case in that figure. For $\eta = 0.499$, the relative inefficiency converges to a very low level. This is not surprising, since the proximity to $\eta = 1/2$ implies that the efficient individual contributions change relatively little regarding increases in population size, and so the equilibrium remains close to the efficient one. When $\eta = 1/4$, the relative inefficiency approaches 0.25, thus the absolute inefficiency is approximately $1/4$ of the ex-ante allocation inefficiency, which is of order greater than I^3, as can be checked from equations (4) and (5). In the case of $\eta = 1$, the inefficiency measure converges to 1 when I goes to infinity, indicating that QF is asymptotically equivalent to the ex-ante provision in terms of efficiency. Lastly, if $\eta = 3/2$, we can see that QF becomes more inefficient than using the ex-ante efficient provision. For values of I close to 200, we observe that the deadweight loss of QF is nearly 4 times larger than that of the ex-ante efficient provision, indicating that in some situations QF can be substantially worse than the second-best provision, which is the provision given by a central planner that has only information regarding the type distribution. The value of $\eta = 1.5$ is often used in the literature in the context of elasticity of consumption (Drupp et al. 2018), and if the same is true in the context of public good consumption, QF might perform worse than our second-best provision level when applied in large populations.

6.2 Changes in the level of uncertainty

To measure the changes in welfare resulting from changes in the level of uncertainty of the model, we consider the next simple two-indiivdual setting. We assume $I = 2$, $|\Theta_1| = 2$, and $|\Theta_2| = 1$. Let $\alpha := \Pr(\theta_1 = \theta_1^1)$ be the probability that individual $i = 1$ has the type θ_1^1. The utility functions for the public good of individual 1, for each of her types, are

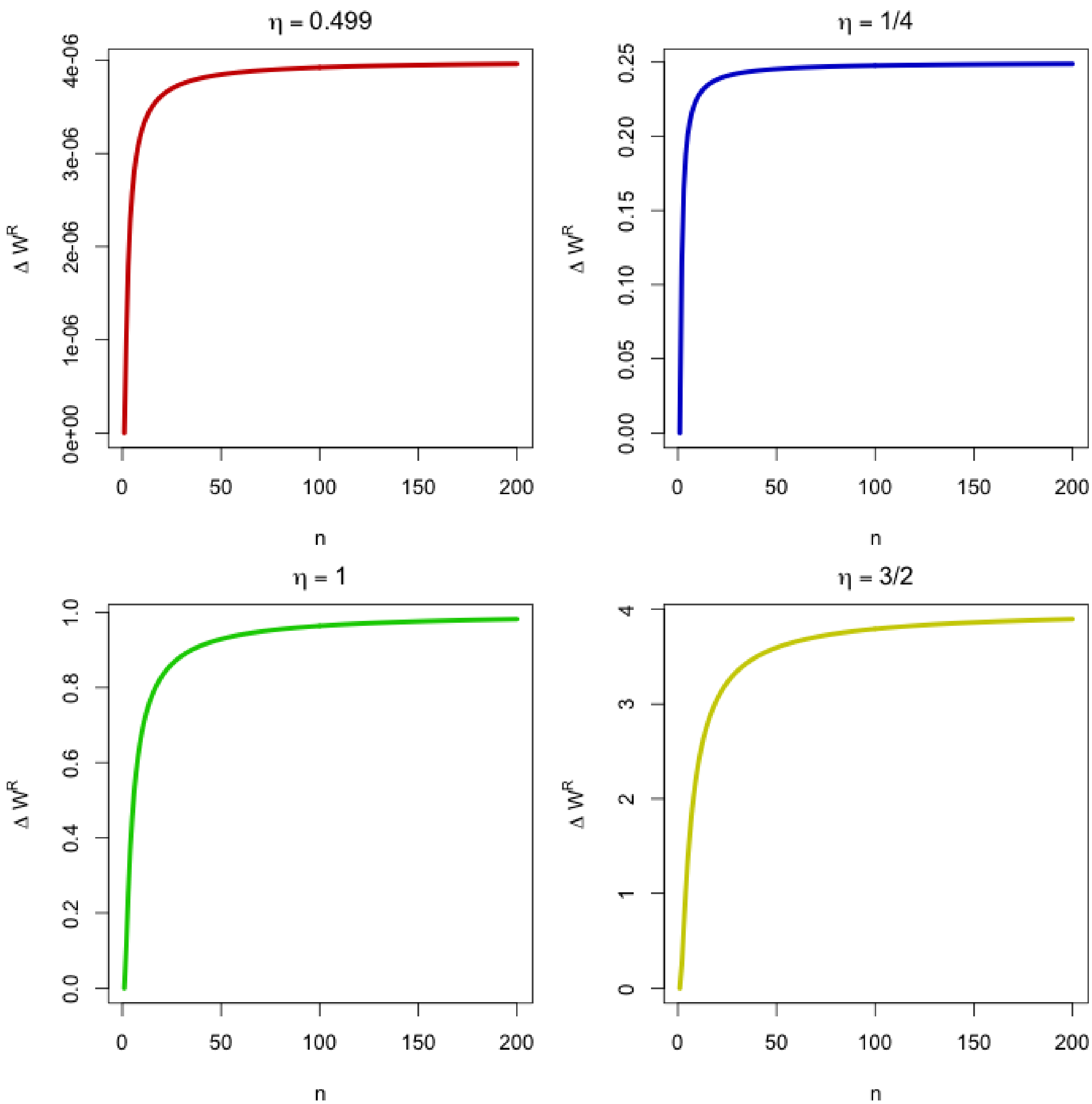

Figure 3: Relative deadweight loss as a function of $1 \leq I \leq 200$.

$$v_1(F;\theta_1^1) = \frac{F^{1-\eta}}{1-\eta},$$

$$v_1(F;\theta_1^2) = \beta_1 \frac{F^{1-\eta}}{1-\eta},$$

and for individual 2,

$$v_2(F;\theta_2^1) = \beta_2 \frac{F^{1-\eta}}{1-\eta},$$

where $\beta_1, \beta_2 \in [0.1, 50]$. We now solve the individual problems. The first individual solves, for each of her types,

$$\max_{c_1 \geq 0} \tilde{\beta}_1 \frac{\left(c_1^{1/2} + c_2(\theta_2^1)^{1/2}\right)^{2(1-\eta)}}{1-\eta} - c_1,$$

where $\tilde{\beta}_1$ is either 1 or β_1 if θ_1^i is either θ_1^1 or θ_1^2, respectively. Solving for each type, it results

$$\left(c_1(\theta_1^1)\right)^{1/2} = \left[\left(c_1(\theta_1^1)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\eta}. \tag{6}$$

$$\left(c_1(\theta_1^2)\right)^{1/2} = \beta_1 \left[\left(c_1(\theta_1^2)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\eta}. \tag{7}$$

Analogously, for individual 2, we have

$$\max_{c_2 \geq 0} \alpha \left(\beta_2 \frac{\left(c_1^{1/2}(\theta_1^1) + c_2^{1/2}\right)^{2(1-\eta)}}{1-\eta}\right) + (1-\alpha)\left(\beta_2 \frac{\left(c_1^{1/2}(\theta_1^2) + c_2^{1/2}\right)^{1-\eta}}{1-\eta}\right) - c_2$$

$$\begin{aligned} \Rightarrow \left(c_2(\theta_2^1)\right)^{1/2} = {} & \alpha\beta_2 \left[\left(c_1(\theta_1^1)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\eta} \\ & + (1-\alpha)\beta_2 \left[\left(c_1(\theta_1^2)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\eta}. \end{aligned} \tag{8}$$

Solving equations (6) to (8) numerically, we obtain the QF equilibrium for this game.

The ex-post efficient level of the public good is the solution of:

$$\max_{F \geq 0} (\tilde{\beta}_1 + \beta_2) \frac{F^{1-\eta}}{1-\eta} - F,$$

which implies that

$$F^e(\theta_1^1, \theta_2^1) = (1 + \beta_2)^{1/\eta}. \tag{9}$$

$$F^e(\theta_1^2, \theta_2^1) = (\beta_1 + \beta_2)^{1/\eta}. \tag{10}$$

Lastly, the ex-ante efficient provision level for the public good solves

$$\max_{F\geq 0} \alpha \left[\frac{F^{1-\eta}}{1-\eta}\right] + (1-\alpha)\left[\beta_1 \frac{F^{1-\eta}}{1-\eta}\right] + \beta_2 \frac{F^{1-\eta}}{1-\eta} - F,$$

from which it results that

$$F^{EA} = (\alpha + (1-\alpha)\beta_1 + \beta_2)^{1/\eta}. \tag{11}$$

For the first group of numerical illustrations, let us fix $\beta_1 = 2$, $\beta_2 = 1$ and consider two alternative values for the elasticity coefficient, $\eta = 1$ and $\eta = 1/4$. Those values of η are chosen because $\eta = 1/2$ is a threshold value, from which individuals have different responses to increases in the other participants' contributions, as we stated in proposition 5.1 for the complete information setting.

Let us start varying α to capture the effect of the degree of uncertainty on welfare. In figure 4, the resulting inverted "U" shape has an intuitive explanation: the closer the game is to complete information ($\alpha = 0$ or $\alpha = 1$) the lower the deadweight loss is, and it is highest at about halfway between these values ($\alpha \approx 0.48$ and $\alpha \approx 0.53$) depending on the value of η.

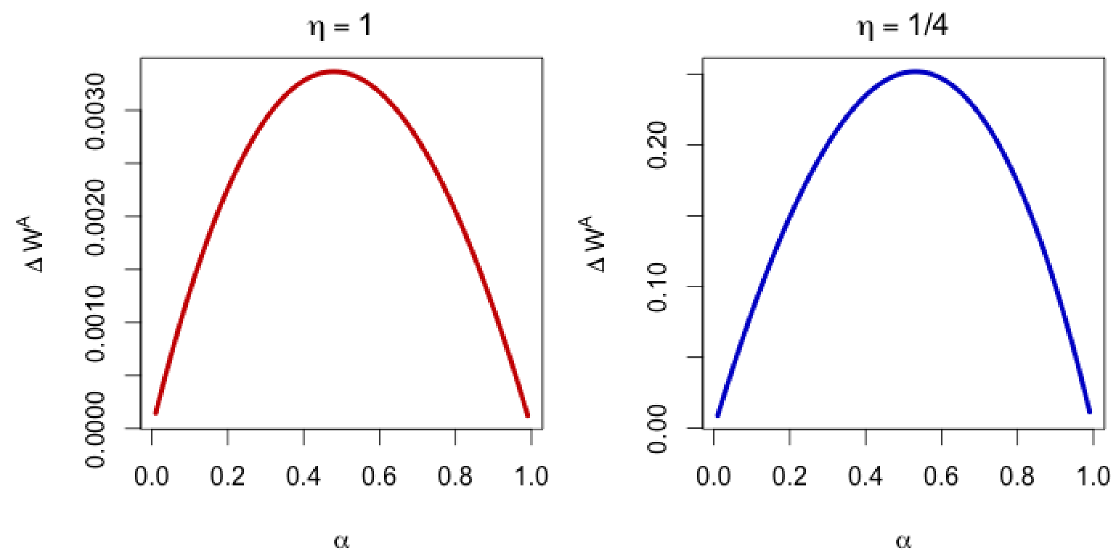

Figure 4: Absolute deadweight loss as a function of $\alpha \in [0, 1]$.

Now, let us analyze the relative deadweight loss when α varies in $(0, 1)$. Figure 5 shows the monotonic response to increases in the probability of individual 1 having lower valuation for the public good. Notice that ΔW^R is not defined for $\alpha = 0$ or $\alpha = 1$, since the ex-ante efficient provision is efficient in those cases. When $\eta = 1$, increases in the probability of individual 1 being of the lower type makes QF less inefficient than the ex-ante efficient provision level. This is compatible with the stabilizing behavior of the best response functions in this case, reported in proposition 5.1. When $\eta = 1/4$, the opposite behavior is observed, increasing the inefficiency with respect to that of the ex-ante provision level. In both cases, QF is considerably more efficient than the ex-ante efficient provision, as the relative deadweight loss ranges between 0.05 and 0.11.

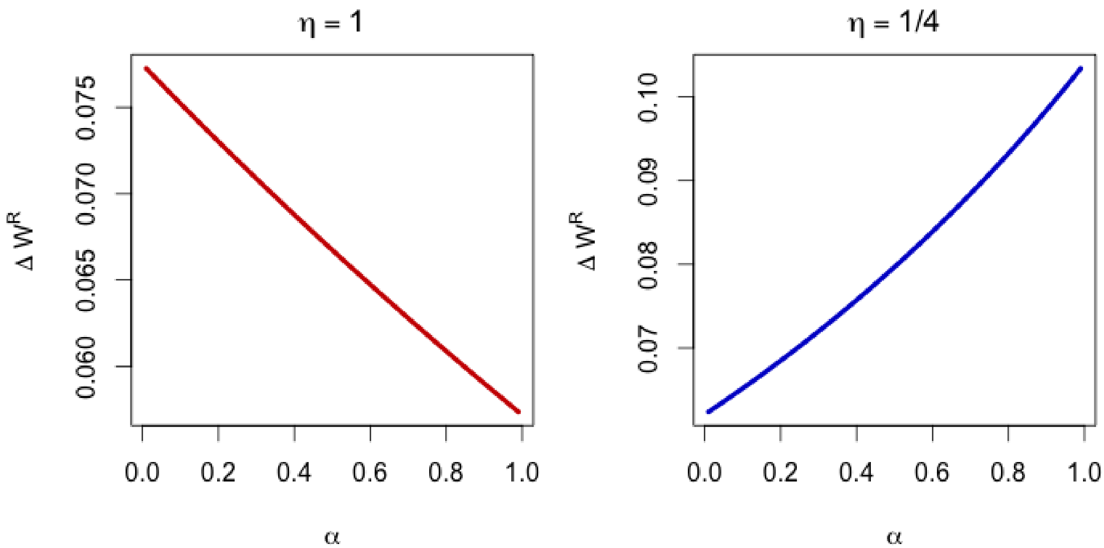

Figure 5: Relative deadweight loss as a function of $\alpha \in (0, 1)$.

Next, we describe the response of welfare to variation in the *intensity* of the incomplete information shock. Namely, fixing the probability of being of type θ_1^1, $\alpha = 1/2$ we vary the value of the shock β_1. In figure 6 we can observe that for both values of η, the absolute deadweight loss increases as the game moves away from the case of complete information ($\beta_1 = 1$). When $\eta = 1 > 1/2$, an increase in β_1 generates an increase in individual 1's contributions (for both types) and a decrease in the individual 2's contribution, as can be checked from equations (6) to (8). As β_1 grows arbitrarily large, the second term on the right-hand side of equation (8) goes to zero, then, individual 2's contribution converges to a positive value, that explains the concave shape of the function for $\beta_1 > 1$. For $\beta_1 < 1$, the same logic implies that individual 2's contribution grows at increasing rates, and so we observe a convex shape of the graph. When $\eta = 1/4 < 1/2$, an analogous reasoning explains the convex shape observed in the figure.

Figure 7 presents the plots for the relative deadweight losses when varying β_1. Since ΔW^R is not defined when $\beta_1 = 1$, we consider the limit of that function as β_1 goes to 1. In both cases ($\eta = 1$ and $\eta = 1/4$), ΔW^R monotonically converges to zero as β_1 goes to infinity. This result indicates that, in a setting of incomplete information, if an individual with complete information has its valuation

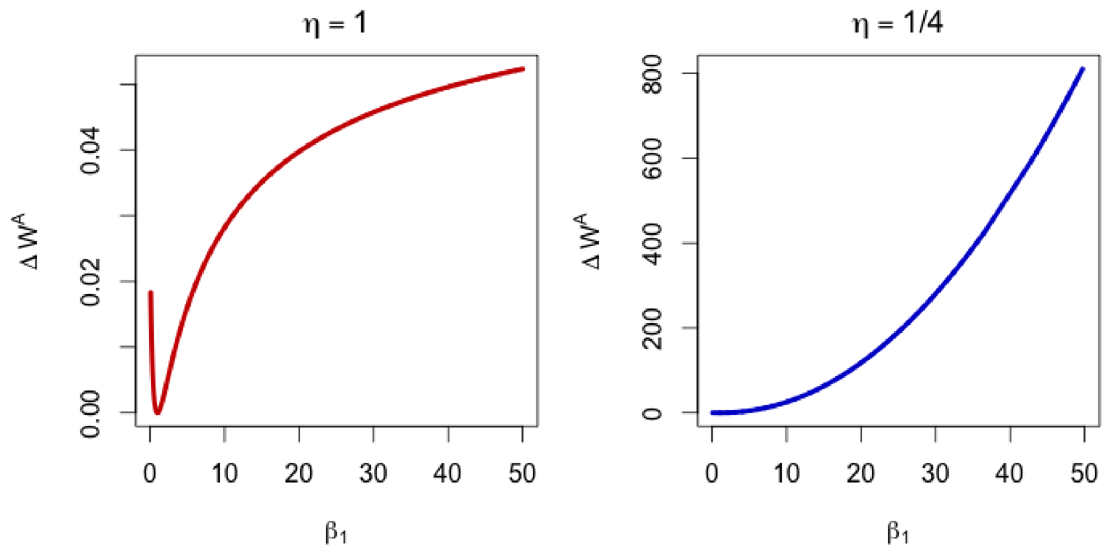

Figure 6: Absolute deadweight loss as a function of $\beta_1 \in [0.1, 50]$.

for the public good increased in any type profile, then the QF provision level becomes relatively more efficient than the ex-ante efficient provision. In other words, the information contained in the contribution chosen by an individual that has complete information under QF becomes more valuable the higher the valuation of this individual for the public good.

In figure 8, we present the relative deadweight loss for changes in β_2 and fixing $\eta = 1.5$. We can see that, as in figure 3, when $\eta > 1$ there are games in which QF is less efficient than the ex-ante efficient provision. Here, we can consider individual 2 to be an aggregation of several indiivduals, so even values of β_2 that are considerably higher than those of β_1 could represent games where there are several indiivduals with a single type and only one with multiple types.

To finalize this subsection, we analyze the impact of variations in the elasticity parameter on the relative deadweight loss. In figure 9 we fix $\alpha = 1/2$ and $\beta_1 = 2$, and vary the value of the elasticity coefficient in the interval $[0.1, 5]$. This interval is compatible with estimates for this parameter presented in a survey of experts (Drupp et al. 2018). We can observe that it approaches efficiency as η goes to $1/2$, just as proposition 5.2 asserted. When η moves away from this value, the inefficiency of QF increases proportionately more than that of the ex-ante level; however, the relative inefficiency remains below the unity, meaning that QF is less inefficient than the second best ex-ante efficient level of funding for the public good.

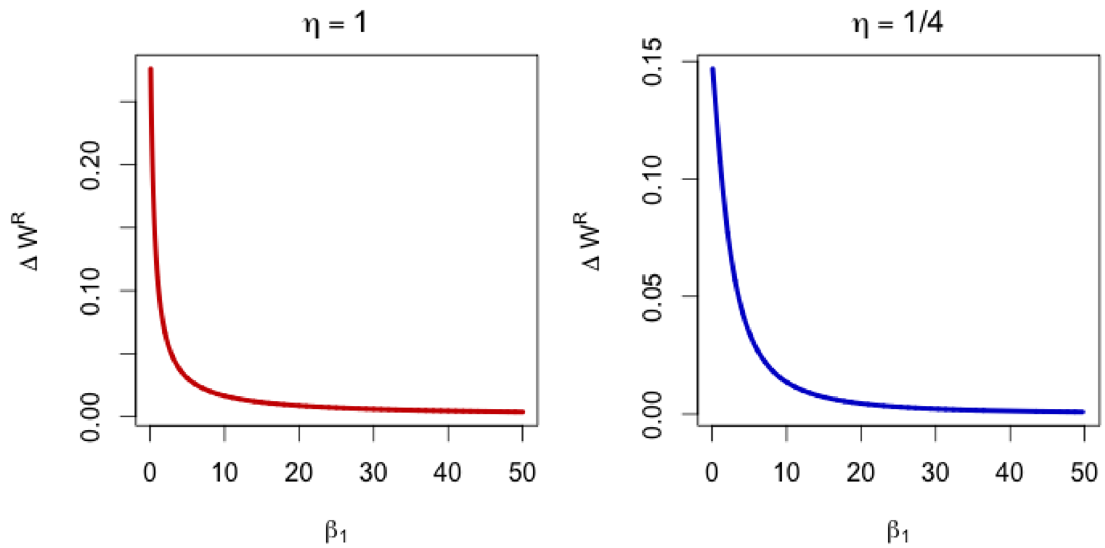

Figure 7: Relative deadweight loss as a function of $\beta_1 \in [0.1, 50]$.

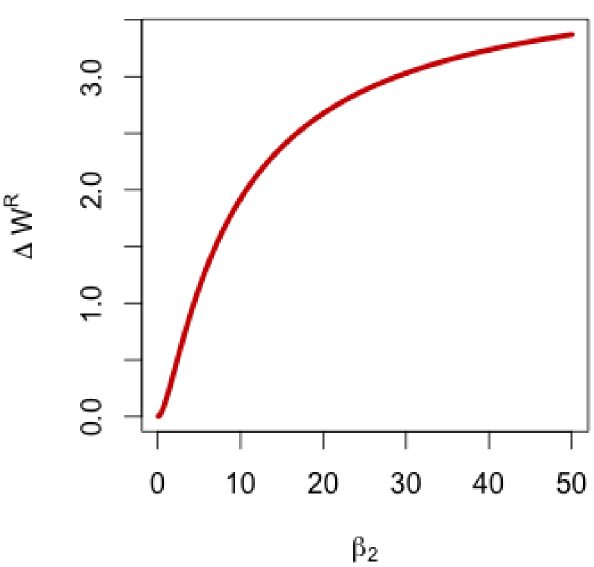

Figure 8: Relative deadweight loss as a function of $\beta_2 \in [0.1, 50]$, for $\eta = 3/2$.

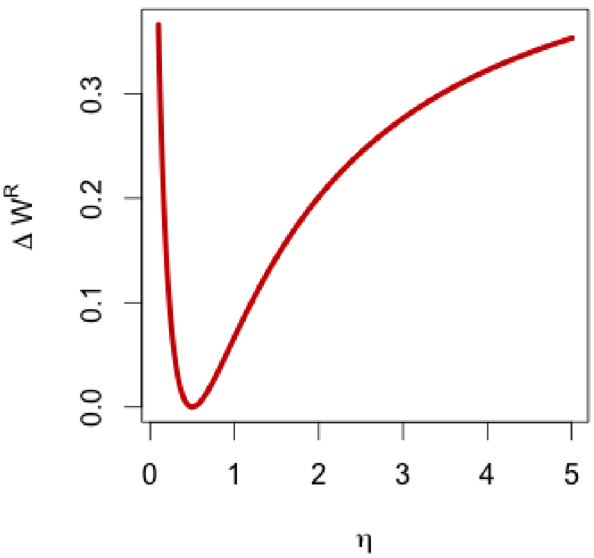

Figure 9: Relative deadweight loss as a function of $\eta \in [0.1, 5]$.

7 Conclusions

Mechanisms to attain the efficiency of decentralized public good provision are widely studied and discussed in the literature. The difficulties posed by additional requirements, such as individual rationality or stability of equilibria, constitute a challenge for theoretical and applied modeling of some of those mechanisms. In this sense, the quadratic funding mechanism arises as a solution for reaching Pareto efficiency in a decentralized way. Its simplicity and efficiency when the efficient amount is strictly positive make it a promising scheme for financing public goods whenever the individuals have complete information regarding their peers' preferences.

In this work, we revisit the quadratic funding mechanism in the complete information setting and analyze the extent to which efficiency is maintained when there is incomplete information. For the complete information framework, we generalize the efficiency result presented by Buterin et al. (2019) to show that efficiency holds in a stronger sense given a standard set of assumptions. On the other hand, in the incomplete information framework, we find that efficiency is only attainable under knife-edge conditions on the utility functions of the participants. In particular, we show that, for the class of isoelastic utility functions representing the participants' preferences for the public good, efficiency results if and only if the elasticity coefficient is equal to 1/2.

In light of these results, we propose two measures for the size of inefficiency for the incomplete information game: the equilibrium's absolute and relative deadweight loss. The absolute (monetary equivalent) measure compares the expected welfare loss of using quadratic funding rather than the ex-post efficient provision. The relative measure is the ratio between the absolute deadweight loss and the expected welfare loss of the ex-ante (second-best) efficient provision, where the second-best allocation is the central planner's provision given that she only knows the distributions of the individuals' types. We utilize a setup where individuals have isoelastic utility functions, and our analysis allows us to conduct several numerical exercises. We find that, first, when the population size increases, absolute deadweight loss asymptotically converges to a low value, possibly zero, when the elasticity coefficient is larger than one. On the other hand, the relative deadweight loss converges to a value larger than one (i.e., the QF allocation is worse than the second-best allocation) when the elasticity coefficient is larger than one. We also show that the effect of changes in the level of uncertainty in the game is also a function of the elasticity coefficient, and whether this coefficient is larger than or smaller than 1/2 has important implications for the overall behavior of these changes.

The results presented here have important implications for the theoretical understanding and practical usage of the quadratic funding mechanism and, more broadly, to quadratic pricing mechanisms. The fact that QF is almost always inefficient under incomplete information presents a crucial restraint to the application of the mechanism for funding public goods. Nevertheless, we have shown that one of the contexts where it is efficient (namely, when the elasticity coefficient is equal to 1/2) corresponds to a situation where individuals do not change their contribution in response to changes in the contributions of others. This constitutes a situation that is both plausible and easily verifiable empirically, and indicates a promising direction for future research. Furthermore, in cases where the mechanism is not efficient, our quantitative estimates allow for an assessment of the inefficiency of the mechanism when estimating the parameters. In particular, if individuals change their contributions in response to changes in the contributions of others, but this change is relatively small, then the mechanism correspondingly leads to results that are close to the optimum; and even for large population sizes, if the elasticity coefficient is smaller than one, then it is superior to our second-best allocation. If this superiority is verified based on empirical estimates of the elasticity coefficient, an important question for future research is how the quadratic funding mechanism performs in practice when compared to commonly used mechanisms, such as majority voting or linear matching rules.

References

Claude Berge. *Topological Spaces*. New York. Macmillan, 1963.

Theodore Bergstrom. When does majority rule supply public goods efficiently? In Steinar Strøm, editor, *Measurement in Public Choice*, pages 75–85. Palgrave Macmillan, 1981. doi: 10.1007/978-1-349-05090-1_6.

Theodore Bergstrom, Lawrence Blume, and Hal Varian. On the private provision of public goods. *Journal of public economics*, 29(1):25–49, 1986.

Vitalik Buterin, Zoë Hitzig, and E. Glen Weyl. A flexible design for funding public goods. *Management Science*, 65(11):5171–5187, 2019.

Edward H. Clarke. Multipart pricing of public goods. *Public Choice*, 11:17–33, 1971.

Moritz A. Drupp, Mark C. Freeman, Ben Groom, and Frikk Nesje. Discounting disentangled. *American Economic Journal: Economic Policy*, 10(4):109–34, 2018. doi: 10.1257/pol.20160240.

Theodore Groves and John Ledyard. Optimal allocation of public goods: A solution to the 'free rider' problem. *Econometrica*, 45(4):783–809, 1977.

Paul J. Healy. Learning dynamics for mechanism design: An experimental comparison of public goods. *Journal of Economic Theory*, 129:114–149, 2006.

Aanund Hylland and Richard Zeckhauser. A mechanism for selecting public goods when preferences must be elicited. 1979.

Louis Kaplow and Scott Duke Kominers. Who will vote quadratically? voter turnout and votes cast under quadratic voting. *Public Choice*, 172(1):125–149, 2017. doi: 10.1007/s11127-017-0412-5.

Steven Lalley and Eric Glen Weyl. Nash equilibria for quadratic voting. SSRN working paper series, https://ssrn.com/abstract=2488763, 2019.

Erik Lindahl. Just taxation - A positive solution. In Peacock A.T. Musgrave R.A., editor, *Classics in the Theory of Public Finance*, pages 168–176. International Economic Association Series. Palgrave Macmillan, London, 1958.

John Milnor. *Topology from Differentiable Viewpoint*. Charlottesville. University Press of Virginia, 1965.

Sunoo Park and Ronald L Rivest. Towards secure quadratic voting. *Public Choice*, 172(1):151–175, 2017.

David Quarfoot, Douglas von Kohorn, Kevin Slavin, Rory Sutherland, David Goldstein, and Ellen Konar. Quadratic voting in the wild: real people, real votes. *Public Choice*, 172(1):283–303, 2017.

Michael H. Rothkopf. Thirteen reasons why the vickrey-clarke-groves process is not practical. *Operations Research*, 55(2):191–197, 2007.

Aldo Rustichini, Mark A. Satterthwaite, and Steven Williams. Convergence to efficiency in a simple market with incomplete information. *Econometrica*, 62(5):1041–1063, 1994.

Paul A. Samuelson. The pure theory of public expenditure. *Review of Economics and Statistics*, 36:387–389, 1954.

Nicolaus Tideman and Florenz Plassmann. Efficient collective decision-making, marginal cost pricing, and quadratic voting. *Public Choice*, 172(1-2):45–73, 2017.

Xavier Vives. Information, strategic behavior, and efficiency in cournot markets. *The RAND Journal of Economics*, 33(3):361–376, 2002.

Mark Walker. A simple incentive compatible scheme for attaining lindahl allocations. *Econometrica*, 49(1):65–71, 1981.

E Glen Weyl. The robustness of quadratic voting. *Public choice*, 172(1):75–107, 2017.

Appendix

Proof. (Proposition 2.1) Let $v : \mathbb{R}_+ \times \Theta \to \mathbb{R}$ be defined by $v(F;\theta) = \sum_{i=1}^{I} v_i(F;\theta_i)$, and fix $\theta \in \Theta$. By Assumption 1, we have that $v(\cdot;\theta) \in C^1$ is strictly concave, and that $\lim_{F\to\infty} v'(F;\theta) = 0$. We then have two cases: $v'(0;\theta) \leq 1$ or $v'(0;\theta) > 1$. In the first case, it follows from the first order condition of definition 5 that $F = 0$ is efficient. Furthermore, since $v(\cdot;\theta)$ is a strictly concave function, we have that $v'(F;\theta) < 1$ for all $F > 0$ and thus, again from definition 5, there can be no efficient provision $F > 0$. Thus, $F^e(\theta) = 0$ is the unique efficient provision.

Now suppose that $v'(0;\theta) > 1$. It follows that $F = 0$ is not efficient. Since $\lim_{F\to\infty} v'(F;\theta) = 0$, there exists $A \in \mathbb{R}$ such that $v'(A;\theta) < 1$. Thus, since $v'(\cdot;\theta)$ is continuous, $v'(0;\theta) > 1$ and

$v'(A;\theta) < 1$, it follows from the intermediate value theorem that there exists $0 < B < A$ such that $v'(B;\theta) = 1$. Additionally, since $v'(\cdot;\theta)$ is strictly decreasing, we have that $v'(F;\theta) \neq 1$ for all $F \neq B$. Therefore, $F^e(\theta) = B$ is the unique efficient provision.

For any $\theta \in \Theta$, we then have that in both cases there is a unique efficient provision $F^e(\theta) \geq 0$. Thus, we construct aunique function $F^e : \Theta \to \mathbb{R}_+$ that maps each profile of types to its efficient funding, as desired.

□

Proof. (Proposition 2.2) We begin by proving the following claims.

Claim A.1. *Suppose that, for all $i \in \mathcal{I}$ and $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i) \in C^1$ is a strictly increasing and strictly concave function. Then, for any $i \in \mathcal{I}$ and $\mathbf{c}_{-i} \in \mathbb{R}_+^{I-1}$, the best-response correspondence $c_i(\cdot;\mathbf{c}_{-i}) : \Theta_i \to \mathbb{R}_+$ is at most single valued.*

Proof. (Claim A.1) Without loss of generality, let us consider $i = 1$. Let $\theta_1 \in \Theta_1$ be the type of this individual, and suppose that there exist $a, b \in \mathbb{R}_+$, $a < b$, best-responses to $\mathbf{c}_{-i}$. Let $\varepsilon = b^{1/2} - a^{1/2} > 0$. From the first order conditions, we have

$$\begin{aligned}
a^{1/2} &= E\left[v_1'\left(\left[a^{1/2} + \sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]^2;\theta_i\right)\cdot\left[a^{1/2} + \sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]\,\middle|\,\theta_i\right] \\
&= a^{1/2}E\left[v_1'\left(\left[a^{1/2} + \sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]^2;\theta_i\right)\,\middle|\,\theta_i\right] \\
&\quad + E\left[v_1'\left(\left[a^{1/2} + \sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]^2;\theta_i\right)\cdot\left[\sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]\,\middle|\,\theta_i\right].
\end{aligned}$$

Notice that the term in the third line above is nonnegative; thus, the equality above is satisfied only if $E\left[v_1'\left(\left[a^{1/2} + \sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]^2;\theta_i\right)\,\middle|\,\theta_i\right] \leq 1$. Thus, since $b > a$ and v_1' is strictly decreasing, we have that

$$\begin{aligned}
&E\left[v_1'\left(\left[b^{1/2} + \sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]^2;\theta_i\right)\cdot\left[\sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]\,\middle|\,\theta_i\right] \\
&< E\left[v_1'\left(\left[a^{1/2} + \sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]^2;\theta_i\right)\cdot\left[\sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]\,\middle|\,\theta_i\right],
\end{aligned}$$

and

$$b^{1/2}E\left[v_1'\left(\left[b^{1/2}+\sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]^2;\theta_i\right)\middle|\,\theta_i\right]$$
$$< a^{1/2}E\left[v_1'\left(\left[a^{1/2}+\sum_{i=2}^{I}(c_i(\theta_i))^{1/2}\right]^2;\theta_i\right)\middle|\,\theta_i\right]+\varepsilon.$$

Thus, adding the inequalities and using the fact that a and b satisfy the first order conditions, it follows that $a^{1/2}+\varepsilon > b^{1/2}$. But this contradicts the definition of ε. This contradiction completes the proof.

Claim A.2. *Suppose that, for all $i \in \mathcal{I}$ and $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i) \in C^1$ is a strictly concave function and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. Then, there exists $A > 0$ such that, for any $i \in \mathcal{I}$ and $\theta_i \in \Theta_i$, if the contributions of all other individuals belong to $[0, A]$ for any profile of types, then i's best-response also belongs to this interval.*

Proof. (Claim A.2) Under the given hypotheses, we can then define, for each $i \in \mathcal{I}$, a function $f_i : \Theta_i \to \mathbb{R}_+$ mapping each θ_i to $f(\theta_i) > 0$ such that $v_i'(f(\theta_i);\theta_i) < 1/I$. Now, letting $A := \max\{f_i(\theta_i); i \in \mathcal{I}, \theta_i \in \Theta_i\}$, by the hypothesis of strict concavity of v_i it follows that $v_i'(A;\theta_i) < 1/I$, for all $i \in \mathcal{I}$ and all $\theta_i \in \Theta_i$. Now, suppose that, for some $j \in \mathcal{I}$, we have that $c_i(\theta_i) \in [0, A]$ for all $i \neq j$ and all $\theta_i \in \Theta_i$, and the best-response of j with type $\theta_j \in \Theta_j$ is $k > A$. Since $k > 0$, the first order conditions for individual j's problem imply that

$$k^{1/2} = E\left[v_j'\left(\left[k^{1/2}+\sum_{i\neq j}(c_i(\theta_i))^{1/2}\right]^2;\theta_j\right)\cdot\left[k^{1/2}+\sum_{i\neq j}(c_i(\theta_i))^{1/2}\right]\middle|\,\theta_j\right]$$
$$< \frac{1}{I}E\left[k^{1/2}+\sum_{i\neq j}(c_i(\theta_i))^{1/2}\,\middle|\,\theta_j\right]$$
$$< \frac{1}{I}I\cdot k^{1/2},$$

where the first inequality follows from $v_j'(A;\theta_j) < 1/I$ and $v_j(\cdot;\theta_j)$ being strictly concave, and the second inequality follows from $c_i(\theta_i) \in [0, A]$ for all $i \neq j$. It results that $k < k$; this contradiction completes the proof.

Now, we return to the proof of the proposition. For each individual $i \in \mathcal{I}$, let $|\Theta_i| = L_i$. We can restrict the domain and range of the best-response functions to the interval $[0, A]$, using $A > 0$ given in Claim A.2. Thus, the problem of individual i with type θ_i is

$$\max_{c_i\in[0,A]} E\left[v_i\left(\Phi^{QF}(c_i,\mathbf{c}_{-i}(\theta_{-i}));\theta_i\right)\,\middle|\,\theta_i\right]-c_i.$$

Notice that the function that is being maximized is continuous and the feasibility correspondence is continuous and compact valued (it is the constant interval $[0, A]$). Thus, by the Theorem of the Maximum (Berge 1963, ch. 6), we have that the best response correspondence for i is not empty and is upper hemicontinuous. Additionally, by Claim A.1, we can conclude that it is a continuous function. Hence, the whole game best-response function $BR : [0, A]^{\sum_{i=1}^{I} L_i} \to [0, A]^{\sum_{i=1}^{I} L_i}$ is also continuous. Since $[0, A]^{\sum_{i=1}^{I} L_i}$ is a compact and non-empty set, the Brouwer fixed point theorem (Milnor 1965) allows us to conclude that BR has a fixed point, which is clearly an equilibrium for QF. □

Proof. (Proposition 3.2) It follows from the assumption of complete information that every individual has a type set with a single element, and that the set of type vectors only has a single element. Thus, to simplify the notation, we use v_i to denote $v_i(\cdot; \theta_i)$, and F^e to denote $F^e(\theta)$.

By proposition 2.1, we know that the hypotheses adopted here guarantee that there is a unique efficient provision $F^e \geq 0$. There are two cases: $F^e = 0$ or $F^e > 0$. First, suppose we have $F^e = 0$. We are going to show that $\mathbf{0}$ is an equilibrium for Φ^{QF}. Consider the problem faced by some individual $i \in \mathcal{I}$ when all other individuals are contributing zero to the mechanism:

$$\max_{c_i \geq 0} v_i \left(\left[c_i^{1/2} + 0 \right]^2 \right) - c_i.$$

Which can be written as

$$\max_{c_i \geq 0} v_i \left(c_i \right) - c_i.$$

Thus, the first order condition for the individual i is that $v_i'(c_i) \leq 1$, with equality holding when $c_i > 0$. But note that, since $F^e = 0$, we have from definition 5 that $\sum_{j=1}^{I} v_j'(0) \leq 1$. In particular, since v_j is increasing for all $j \in \mathcal{I}$, this implies that $v_i'(0) \leq 1$. Thus, $c_i = 0$ satisfies the first order condition for i. But since the choice of i was arbitrary, we have that $\mathbf{0}$ is an equilibrium for Φ^{QF}.

Now, suppose $F^e > 0$. For all $i \in \mathcal{I}$, let

$$c_i = \left(v_i'(F^e) \cdot (F^e)^{1/2} \right)^2. \tag{12}$$

It is easy to check that $F^e = \Phi^{QF}(\mathbf{c}) = [\sum_{i=1}^{I} c_i^{1/2}]^2$. Rearranging, we obtain

$$v_i'(F^e) = \frac{(c_i)^{1/2}}{(F^e)^{1/2}},$$

which is precisely the first order condition for the individual i's optimization problem in the QF mechanism. We can thus conclude that the vector $\mathbf{c} = (c_1, \cdots, c_I)$ as defined by equation (12) is an equilibrium of QF and its provision is efficient.

Thus, in all cases, there exists an equilibrium allocation $\mathbf{c}^*$ such that $\Phi^{QF}(\mathbf{c}^*) = F^e$, as we wanted to show.

□

Proof. (Proposition 4.1) First, suppose that $F^e(\theta) = 0$ for all $\theta \in \Theta$. Let us prove that $\mathbf{c}(\theta) = 0$ for all θ is an equilibrium for QF. Suppose that, for some $i \in \mathcal{I}$ we have that all other individuals are playing the strategy profile $\mathbf{c}_{-i}(\theta_{-i}) = 0$. The problem faced by the individual i with some type $\theta_i \in \Theta_i$ is given by

$$\max_{c_i \geq 0} E\left[v_i\left(\Phi^{QF}(c_i, \mathbf{c}_{-i}(\theta_{-i})); \theta_i\right) \,\middle|\, \theta_i\right] - c_i.$$

Substituting $\mathbf{c}_{-i}(\theta_{-i}) = 0$ and the definition of QF, the problem above becomes

$$\max_{c_i \geq 0} v_i\left(c_i; \theta_i\right) - c_i,$$

let $c_i(\theta_i)$ be the solution, then the first order conditions are

$$v_i'(c_i(\theta_i); \theta_i) \leq 1,$$

with equality holding if $c_i(\theta_i) > 0$. Since $F^e(\theta) = 0$ for all $\theta \in \Theta$, it implies that $\sum_{j=1}^{I} v_j'(0; \theta_j) \leq 1$. It follows that $v_i'(0; \theta_i) \leq 1$. Thus, $c_i(\theta_i) = 0$ satisfies the first order conditions for i, as we wanted to show.

Now, we prove the reciprocal using a contradiction argument. The hypothesis asserts that there exist $\theta' \in \Theta$ for which $F^e(\theta') = 0$ and that QF is efficient. Then, suppose that there is a $\theta'' \in \Theta$, such that $F^e(\theta'') > 0$. Let $\mathbf{c}$ be an efficient equilibrium for QF. The first order condition for individual i's problem for the type profile θ'' is

$$E\left[v_i'(\Phi^{QF}(\mathbf{c}^*(\theta)); \theta_i'') \cdot \left(\frac{\Phi^{QF}(\mathbf{c}^*(\theta))}{c_i^*(\theta_i'')}\right)^{1/2} \,\middle|\, \theta_i''\right] \leq 1, \tag{13}$$

which cannot be satisfied for $c_i^*(\theta_i'') = 0$, since $\Pr(\theta'') > 0$, $F^*(\theta'') = F^e(\theta'') > 0$, and $v_i'(F^*(\theta''); \theta_i'') > 0$. Therefore, $c_i^*(\theta_i'') > 0$. Since i is arbitrary, it results $\mathbf{c}^*(\theta'') \gg \mathbf{0}$. Now, let us consider the first order condition of individual i's problem when her type is θ_i',

$$E\left[v_i'(\Phi^{QF}(\mathbf{c}^*(\theta)); \theta_i') \cdot \left(\frac{\Phi^{QF}(\mathbf{c}^*(\theta))}{c_i^*(\theta_i')}\right)^{1/2} \,\middle|\, \theta_i'\right] \leq 1, \tag{14}$$

with equality holding when $c_i^*(\theta_i') > 0$. Since $\Pr(\theta_i', \theta_{-i}'') > 0$, and $\mathbf{c}_{-i}^*(\theta_{-i}'') \gg \mathbf{0}$, we have that $c_i^*(\theta_i') > 0$, because the numerator of the expected value when $\theta = (\theta_i', \theta_{-i}'')$ is strictly positive. Thus, $c_i^*(\theta_i') > 0$. But then, $\Phi^{QF}(\mathbf{c}(\theta')) > 0 = F^e(\theta')$, which is a contradiction to the efficiency of QF. This completes the proof. □

Proof. (Proposition 4.2) Without loss of generality, let $j = 1$. First, suppose that QF is efficient, that is, there exists an equilibrium strategy profile $\mathbf{c}^*$ such that $\Phi^{QF}(\mathbf{c}^*(\theta)) = F^e(\theta)$. For $\theta \in \Theta$, the efficiency of $F^e(\theta) > 0$ implies

$$v_1'(F^e(\theta); \theta_1) + \sum_{i=2}^{I} v_i'(F^e(\theta); \theta_i^1) = 1.$$

Thus,

$$\sum_{i=2}^{I} v_i'(F^e(\theta); \theta_i^1) = 1 - v_1'(F^e(\theta); \theta_1). \tag{15}$$

Since $j = 1$ has complete information regarding the other individuals' preferences, it follows that

$$v_1'(F^e(\theta); \theta_1) = \frac{(c_1(\theta))^{1/2}}{(F^e(\theta))^{1/2}}. \tag{16}$$

Thus, equations (15) and (16) imply that

$$\sum_{i=2}^{I} v_i'(F^e(\theta); \theta_i^1) = \frac{\sum_{i=2}^{I}(c_i(\theta_i^1))^{1/2}}{(F^e(\theta))^{1/2}},$$

and so letting $A = \sum_{i=2}^{I}(c_i(\theta_i^1))^{1/2}$ yields the desired result.

To prove the converse, suppose that there exists a constant $A \in \mathbb{R}$ such that $\sum_{i=2}^{I} v_i'(F^e(\theta); \theta_i^1) = A(F^e(\theta))^{-1/2}$. Define the contribution of individual $i \in \mathcal{I}$ with type $\theta_i \in \Theta_i$ by

$$c_i(\theta_i) = E\left(\left[v_i'(F^e(\theta)) \cdot (F^e(\theta))^{1/2} \,\middle|\, \theta_i^1\right]\right)^2, \tag{17}$$

which is well-defined since $F^e(\theta)$ exists and is unique. Note that these contributions satisfy the first order conditions if the generated level of the public good is equal to $F^e(\theta)$, so let us prove this equality.

Since each individual $i = 2, \cdots, n$ has only one single type, the conditional expectation in equation (17) is equal to the unconditional expectation. Taking the square root of both sides and taking the sum yields

$$\begin{aligned}
\sum_{i=2}^{I}(c_i(\theta_i^1))^{1/2} &= \sum_{i=2}^{I} E\left[v_i'(F^e(\theta)) \cdot (F^e(\theta))^{1/2}\right] \\
&= E\left[(F^e(\theta))^{1/2} \sum_{i=2}^{I} v_i'(F^e(\theta))\right] \\
&= E\left[(F^e(\theta))^{1/2} \frac{A}{(F^e(\theta))^{1/2}}\right] \\
&= A.
\end{aligned}$$

Thus, substituting the left-hand side on $\sum_{i=2}^{I} v_i'(F^e(\theta);\theta_i^1) = A(F^e(\theta))^{-1/2}$ we get

$$\sum_{i=2}^{I} v_i'(F^e(\theta);\theta_i^1) = \frac{\sum_{i=2}^{I}(c_i(\theta_i^1))^{1/2}}{(F^e(\theta))^{1/2}} \tag{18}$$

On the other hand, since the individual 1 has complete information, rearranging equation (17) we get

$$v_1'(F^e(\theta);\theta_1) = \frac{(c_1(\theta_1))^{1/2}}{(F^e(\theta))^{1/2}}. \tag{19}$$

Finally, adding equations (18) and (19), and using the fact that $\sum_{i=1}^{I} v_i'(F^e(\theta);\theta_i) = 1$ (because $F^e(\theta)$ is the efficient provision), we get that $\Phi^{QF}(\mathbf{c}(\theta)) = F^e(\theta)$. Thus, it follows that the contributions specified in equation (17) do indeed satisfy the first order conditions of an equilibrium, and that this equilibrium is efficient, as we wanted to show.

□

Proof. (Proposition 5.2) If $\eta = 1/2$, then for all $i \in \mathcal{I}$ we have that $v_i'(F;\theta_i) = \beta_i(\theta_i)F^{-1/2}$. For every $i \in \mathcal{I}$, let us define $c_i(\theta_i)$ by

$$c_i(\theta_i) = \left(E\left[v_i'(F^e(\theta);\theta_i)\cdot(F^e(\theta))^{1/2}\,\middle|\,\theta_i\right]\right)^2, \tag{20}$$

which is well-defined since $F^e(\theta)$ exists and is unique. These contributions satisfy the first order conditions of the individual problem. Let us show that $\Phi^{QF}(\mathbf{c}(\theta)) = F^e(\theta)$. Substituting $v_i'(F;\theta_i)$ in equation (20) we get

$$\begin{aligned}(c_i(\theta_i))^{1/2} &= E\left[\beta_i(\theta_i)\cdot(F^e(\theta))^{-1/2}\cdot(F^e(\theta))^{1/2}\,\middle|\,\theta_i\right]\\ &= \beta_i(\theta_i),\end{aligned}$$

therefore, we have

$$v_i'(F^e(\theta);\theta_i) = \frac{(c_i(\theta_i))^{1/2}}{(F^e(\theta))^{1/2}}.$$

So, adding this expression for all i,

$$\sum_{i=1}^{I} v_i'(F^e(\theta);\theta_i) = \frac{\sum_{i=1}^{I}(c_i(\theta_i))^{1/2}}{(F^e(\theta))^{1/2}}.$$

Hence, since $F^e(\theta)$ is efficient, we have that the left-hand side of the above equation is equal to one. We then have that $\Phi^{QF}(\mathbf{c}(\theta)) = \left(\sum_{i=1}^{I}(c_i(\theta_i))^{1/2}\right)^2 = F^e(\theta)$, and the equilibrium $\mathbf{c}$ is efficient.

To prove the converse, suppose that QF is efficient and $\eta \neq 1/2$. By hypothesis, we know that there exists some $j \in \mathcal{I}$ and types $\theta_j^k, \theta_j^\ell \in \Theta_j$ such that $\beta_j(\theta_j^k) \neq \beta_j(\theta_j^\ell)$. Without loss of generality, let $\beta_j(\theta_j^k) > \beta_j(\theta_j^\ell)$. Now, let $\overline{\theta} \in \Theta$ be the type profile such that $\beta_i(\overline{\theta}_i) \geq \beta_i(\theta_i)$, for all $i \in \mathcal{I}$ and all $\theta_i \in \Theta_i$. Let $\mathbf{c}^*$ be the efficient equilibrium. Then, the first order condition for the individual i in the type profile $\overline{\theta}$ imply that

$$(c_i^*(\overline{\theta}_i))^{1/2} = \beta_i(\overline{\theta}_i) \cdot E\left[(F^e(\theta))^{1/2-\eta} \,\middle|\, \overline{\theta}_i\right]. \tag{21}$$

Let us suppose that $\eta > 1/2$ (the case $\eta < 1/2$ is analogous). First, efficiency implies that, for any $\theta \in \Theta$, $\sum_{i=1}^I v_i'(F^e(\theta);\theta_i) = \sum_{i=1}^I \beta_i(\theta_i) \cdot (F^e(\theta))^{-\eta} = 1$. Then, as $\beta_i(\overline{\theta}_i) \geq \beta_i(\theta_i)$ for all i and all θ_i, it follows that $\sum_{i=1}^I v_i'(F^e(\theta);\overline{\theta}_i) = \sum_{i=1}^I \beta_i(\overline{\theta}_i) \cdot (F^e(\theta))^{-\eta} \geq 1$, for all $\theta \in \Theta$. From strict monotonicity of $v_i'(\cdot;\theta_i)$, we have that $F^e(\overline{\theta}) \geq F^e(\theta)$, for all $\theta \in \Theta$, in particular, $F^e(\overline{\theta}) > F^e(\theta_j^\ell, \overline{\theta}_{-j})$. Thus, as $1/2 - \eta < 0$, it follows that $(F^e(\overline{\theta}))^{1/2-\eta} \leq (F^e(\theta))^{1/2-\eta}$ for all $\theta \in \Theta$ and, in particular, $(F^e(\overline{\theta}))^{1/2-\eta} < (F^e(\theta_j^\ell, \overline{\theta}_{-j}))^{1/2-\eta}$. Since $\Pr\left(\theta_j^\ell \,\middle|\, \overline{\theta}_i\right) > 0$ for all $i \neq j$, we then have

$$E\left[F^e(\theta))^{1/2-\eta} \,\middle|\, \overline{\theta}_i\right] \geq (F^e(\overline{\theta}))^{1/2-\eta} \tag{22}$$

for all $i \in \mathcal{I}$, with strict inequality when $i \neq j$. By (21) and (22) we have that

$$(c_i^*(\overline{\theta}_i))^{1/2} \geq \beta_i(\overline{\theta}_i) \cdot (F^e(\overline{\theta}))^{1/2-\eta} \tag{23}$$

for all $i \in \mathcal{I}$, with strict inequality when $i \neq j$. Adding (23) across all $i \in \mathcal{I}$ yields

$$\sum_{i=1}^I (c_i^*(\overline{\theta}_i))^{1/2} = (F^e(\overline{\theta}))^{1/2} > \sum_{i=1}^I \beta_i(\overline{\theta}_i) \cdot (F^e(\overline{\theta}))^{1/2-\eta}. \tag{24}$$

Dividing both sides of the inequality in (24) by $(F^e(\overline{\theta}))^{1/2}$, we get

$$1 > \sum_{i=1}^I \beta_i(\overline{\theta}_i) \cdot (F^e(\overline{\theta}))^{-\eta} = \sum_{i=1}^I v_i'(F^e(\overline{\theta}); \overline{\theta}_i), \tag{25}$$

which is a contradiction to the definition of efficient provision. This completes the proof. $\square$