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GLOBAL PRIORITIES INSTITUTE

UNIVERSITY OF OXFORD

Dynamic Public Good Provision under Time Preference Heterogeneity

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Abstract

I explore the implications of time preference heterogeneity for the private funding of public goods. The assumption that players use a common discount rate is knife-edge: relaxing it yields substantially different equilibria, for two reasons. First, time preference heterogeneity motivates intertemporal polarization, analogous to the polarization seen in a static public good game. In the simplest settings, more patient players spend nothing early in time and less patient players spending nothing later. Second, and consequently, time preference heterogeneity gives less patient players a "first-mover advantage". Departures from the common-discounting assumption are economically significant: a patient player's payoff in equilibrium, relative to that obtained when he is constrained to act according to a higher discount rate, typically grows unboundedly as his share of the initial budget falls to zero. Finally I discuss applications of these results to the debate over legal disbursement minima.

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1 Introduction

1.1 Motivation

The modern literature on public good games begins with Bergstrom et al. (1986). Bergstrom et al. observed that well-behaved static public good games, with each player's payoff concave in the total provision of each good, feature a unique equilibrium provision of each good. In any equilibrium, each individual is indifferent to marginal reallocations of resources among the goods she herself funds and weakly prefers reallocations to goods she funds from goods she does not. Even individuals with relatively similar preferences thus typically find themselves "polarized", in the sense that they fund entirely or almost entirely non-overlapping sets of projects. Similar observations have been made independently on other occasions, e.g. by Kalai and Kalai (2001).

An extensive literature on dynamic public good games has developed as well. A central concern of this literature is efficiency, and with minor variations, the concern is typically explored in models with a single public good, a single private good, and symmetric players. It has long been understood that in a simple continuous-time model with (near-)perfect monitoring of the individual or aggregate contribution history—or, equivalently, in a discrete-time model with lagged monitoring but (near-)fully patient players—(near-)fully efficient contribution schedules can typically obtain in subgame perfect equilibrium (McMillan, 1979; Benhabib and Radner, 1992; Battaglini et al., 2014). This is because the gains from deviation are infinitesimal relative to the losses from future punishments. At the same time, dynamic interaction can introduce channels that exacerbate crowding out relative to a static setting, in letting players signal that they will contribute little in total by having contributed little in the past (Fershtman and Nitzan, 1991; Admati and Perry, 1991). Conversely, but by similar reasoning, if the public good exhibits increasing returns, a dynamic interaction can allow players "crowd each other in" (Marx and Matthews, 2000). In all cases, the dynamic setting interacts with strategic polarization only in the sense that it can affect how one player's expenditures on the public good crowd or fail to crowd others' out.

Relatedly, despite its size, the literature on dynamic public good games near-universally assumes that the actors under consideration act under a common discount rate. This assumption is pervasive both in the purely theoretical literature, as cited above, and in applications, such as to country-level efforts to mitigate climate change: see e.g. Dutta (2017). Fishman (2019) explores bargaining over public good provision in a dynamic setting where the players use different discount rates, and a small literature building on Sorger (2006) likewise explores the implications of bargaining under time preference heterogeneity in dynamic settings, some of which could apply to public good provision problems. Finally, the only other paper on the theory of public good games even to mention time preference heterogeneity, of which I am aware—and so the only paper to do so outside a bargaining context—is Jacobsen et al. (2017), but it is set in a static environment. Lower time preference, i.e. greater concern for the future, is simply listed as one reason why individuals may have different preferences regarding the provision of an environmental good in the present.

This paper introduces time preference heterogeneity among public good providers, and argues that a natural and important implication is "intertemporal polarization", or crowding out across time.

An analysis of dynamic public good provision under time preference heterogeneity is valuable for at least two reasons.

First, individual rates of time preference vary widely.[1] Most developed-world governments publish discounting guidelines that make explicit the discount rates they use in cost-benefit analysis for public policy, revealing unambiguously that they too act under heterogeneous rates of time preference.[2] Economists' recommendations of time preferences to use in social discounting differ substantially.[3] Philanthropists' time preferences appear to vary as well, both with each other and with those of individuals and policymakers.[4] Of course, individuals, policymakers, and philanthropists all regularly contribute to public goods to which other such parties also contribute, and these parties must all decide how to allocate their contributions over time. In doing so, they participate in dynamic public good games. Real-world dynamic public good games therefore likely exhibit substantial time preference heterogeneity.[5] Our attempts to model these games, and improve public good provision processes in light of them, will likely fail if we do not account for it.

Second, in practice, many individuals currently hold philanthropically-purposed

[1]A dated but helpful review of econometric and experimental literature on time preference heterogeneity among individuals and households can be found in Alan and Browning (2010), pp. 1252–3.

[2]Compare US Office of Management and Budget (2019) and HM Treasury (2020), for instance. This is especially relevant in the context of international contributions to global public goods, because, in the absence of strong international governance, nations must effectively engage in public good games.

[3]As surveyed by Drupp et al. (2018).

[4]Elsey and Moss (2024) survey philanthropists and small donors in the Effective Altruism community on their philanthropic preferences. Of 377 respondents who answered a question about time preference, 44% report a zero rate of pure time preference and 10% report a rate of 5% per year or more. This survey question was included at my request.

[5]Note in particular that philanthropy often involves providing consumption goods to others who also to some extent provide for themselves. Time preference heterogeneity is almost intrinsic to the interaction between these two providers. This is because beneficiaries typically face mortality risks, and temptations to impatience, that we should not generally expect to appear—and which, to some extent, empirically do not appear—in the utility function of a third-party provider. See Nesje (2024) for a model grounding this difference between households' and planners' discount rates, and Frietas-Groff and Makkar (2023) for an experimental approach on this question and review of the relevant literature.

assets in tax-exempt vehicles where they are earmarked for future charitable giving, such as donor-advised funds ("DAFs"). Assets in DAFs in particular, in the United States, currently total almost $150 billion; contributions to them have historically grown at a substantially higher rate than charitable contributions as a whole; and disbursements have not risen as quickly as contributions (National Philanthropic Trust, 2020). As we will see, this pattern can straightforwardly be explained as rational behavior by patient philanthropists given "over-spending" (from their perspective) by less patient other parties. Nevertheless, it is routinely criticized as an unjustifiable withholding of charitable funds, or even as a form of tax evasion. These criticisms recently reached new prominence in the United States with the June 2021 introduction of the Accelerating Charitable Efforts ("ACE") Act by Senators Angus King and Charles Grassley, which would impose disbursement requirements on DAFs, effectively requiring their contributors to act less patiently. Due to a lack of literature on dynamic public good provision under time preference heterogeneity, the implications of such a requirement, and of similar proposals to introduce or raise charitable disbursement minima in the United States and elsewhere, have not undergone thorough economic scrutiny.

I consider a simple model in which there is a single public good, and each player's flow utility depends only on total flow spending on the good. Two players with constant but different rates of time preference make decisions about how much to contribute to the good, and how much to invest for future contribution, over an infinite horizon in continuous time.

To isolate the implications of time preference heterogeneity, I assume that the present value of each player's total contribution is fixed. Contributors decide only the schedule on which to deploy their spending, not the extent to which they will spend on public as opposed to private goods each period. As a result, the model does not resemble the existing literature on dynamic public good provision so much as Bergstrom et al.'s original paper. The relevant change is that I explore what happens when individuals choose their contribution levels for an infinite stream of public goods over which their respective preferences differ—i.e. funding at t, for all $t \geq 0$—*in sequence*, rather than simultaneously.

The model, despite its simplicity, allows us to draw some important and broad conclusions about the implications of heterogeneous discounting for public good provision.

First, the common discounting assumption is knife-edge: even slight time preference differences, even (indeed especially) by small players, give rise to very different equilibria. In particular, time preference differences generate uniquely simple equilibria in which spending is "polarized" in the sense above, with the impatient exclusively responsible for public good funding before some date the patient exclusively responsible after. Furthermore, such equilibria are "asymmetric": they give impatient players a Stackelberg-like first-mover advantage with no analogue in a common-discounting setting.

Second, time preference heterogeneity is highly payoff-relevant. The equilibria that obtain under heterogeneous discounting can offer payoffs, at least for unusually patient parties, which differ dramatically from the payoffs they achieve when constrained to spend as would seem optimal in the absence of time preference heterogeneity.

1.2 Related discounting literature

Though time preference heterogeneity has not been studied in the setting of dynamic public good provision by private funders, three strands of literature have studied time preference heterogeneity in adjacent settings. The results of the present paper prove complementary to the central findings in each of these settings.

One strand concerns the collective allocation of private consumption over time—i.e., under certain preference aggregability assumptions, the discounting behavior of a representative agent—in a population of households with heterogeneous time preferences. A classic observation from this literature is that consumption (Rader, 1981) and/or wealth (Becker, 1980; Ryder, 1985) can, in the limit, become entirely concentrated in the hands of society's most patient members simply because they consume less and invest more. Another observation, closely related to the first, is shown in an exchange economy by Gollier and Zeckhauser (2005) and with variations elsewhere: that given complete markets, a representative agent, if one exists, will exhibit a discount rate that declines with time to that of society's most patient members. In other words, interest rates fall as patient parties lend to their less patient counterparts and command an ever-growing share of the financial market. Heal and Millner (2014) argue that policymakers aiming to set discounting policy for the provision of a public good, while deferring to the time preferences of their constituents, do best to defer to this aggregated discounting schedule. As we will see, similar dynamics play out among agents spending on public goods directly, and the strategic logic of intertemporal polarization can make the tendency for wealth to concentrate in patient hands—and the corresponding tendency of a representative agent to grow more patient with time—even more extreme.

A second body of relevant research concerns optimal taxation by policymakers more patient than their constituents. Farhi and Werning (2007, 2010) analyze optimal taxation in an intergenerational model where individuals save insufficiently, from the patient social planner's perspective, for their descendants. Household consumption in these models is a public good: its provision satisfies the preferences of multiple parties (the policymaker and the household itself) nonexcludably and nonrivally. Similarly, von Below (2012), Belfiori (2017), and Barrage (2018) study optimal carbon taxation and/or investment subsidization in contexts where present production confers both future costs and future benefits (from climate damage and capital accumulation respectively). An important lesson from this literature is that patient policymakers might like to invest resources for future spending, but that to avoid crowding out private investment, it is often optimal for them instead to subsidize private investment and tax private consumption.

Time preference heterogeneity has different implications in the context of optimal taxation than in the context of private spending on public goods, however. The former setting involves an asymmetry in the players' strategy sets: households cannot tax or subsidize policymakers, but policymakers can tax and subsidize households. At least in the absence of political or informational constraints, policymakers endorsing a given time preference rate can often use these tools to implement population-wide behavior that is optimal or near-optimal from the policymaker's perspective.

Finally, a small literature considers public good provision over time by committees of social planners with different time preference rates. One finding is that any attempt to define and implement "optimal" spending plans by such a committee faces some variety of the preference aggregation impossibilities faced in other social choice contexts, as explored in detail by Chambers and Echenique (2018). Millner (2020) proposes a method by which the discounting planners might reach a kind of consensus, but such proposals are themselves inevitably vulnerable to disagreement. Another concern of the literature is that, once an optimality criterion or consensus has been reached, the resulting spending plans are typically time inconsistent. In particular, Jackson and Yariv (2015) show that any social welfare function used in this setting must be either dictatorial or time inconsistent, in that future committee meetings will, if they use the same forward-looking social welfare function, decide to revise the plans made by previous meetings—at least if these were made naively, without taking the possibility of future revisions into account. Millner and Heal (2018) therefore examine the collective decision-making of discounting committees aware that they are playing a dynamic game with their future selves. They find that attempts to implement weighted utilitarian social discounting in such a dynamic game will generally be inefficient. By contrast, I find that decentralized private actors strategically allocating public good contributions over time can implement efficient, weighted utilitarian social discounting.

1.3 Outline

The structure of this paper is as follows.

Section 2 opens with three benchmark settings to which the dynamic public good game is later compared. The first is that of dynamic private good provision. The second and third are static transformations of a dynamic public good game: an "open-loop" setting in which the parties simultaneously commit to a spending schedule, and a "Stackelberg" setting in which the less patient player commits to a spending schedule and the patient player then chooses his best response. The former illustrates the intertemporal polarization, and the latter illustrates both the intertemporal polarization and the first-mover advantage, observed in the dynamic game.

In Section 3, I explore the dynamic game. I find that it has many equilibria, including efficient equilibria, but only one that is polarized; and that this polarized equilibrium is also the only equilibrium that is continuous with equilibrium behavior over a lengthening finite horizon. It implements the profile of spending schedules observed in the equilibrium of the static Stackelberg game.

In Section 4, I introduce constraints in the form of a minimum and/or maximum on the proportional rate at which a player may spend. I then illustrate the economic significance of time preference heterogeneity by calculating a player's willingness to pay to move from the maximally polarized equilibrium of the constrained game to the polarized equilibrium of the unconstrained game of Section 3. I find that, for an atypically patient player (but not for an atypically impatient player), this willingness to pay approaches the entirety of his budget as his budget share—his fraction of the sum of the parties' budgets—goes to zero. That is, when most of the funding for his chosen cause is governed impatiently, a patient player finds a disbursement minimum approximately as costly as a total expropriation.

Finally, Section 5 briefly shows that intertemporal polarization is not an artefact of the simple preference specification of the games above, in which each player's flow utility is isoelastic in total flow spending, but holds under much weaker preference conditions. Since intertemporal polarization largely drives the subsequent results (on the impatient first-mover advantage and the costs of spending constraints), this suggests that the conclusions of Section 3 and Section 4 are relatively robust.

Section 6 concludes.

2 Benchmarks

To build our intuitions about the implications of time preference heterogeneity among public good funders, we consider three benchmarks.

First, in Section 2.1, we review the implications of time preference heterogeneity among households purchasing private goods, making the standard observation that optimal spending rates rise (i.e. optimal saving rates fall) continuously with rates of time preference.

In Section 2.2, we study a static game in which two players with different discount rates simultaneously set schedules on which they will spend on a public good. (It is the "open loop" transformation of the dynamic public good game studied in the next section.) This game has a unique and fully polarized Nash equilibrium, with only the less patient player spending early in time and only the more patient spending later in time. It illustrates how time preference differences can motivate more extreme differences in saving behavior in the context of public good funding than in the context of private good funding.

The last benchmark, in Section 2.3, is a two-period game, like the static game above but with the less patient player as Stackelberg leader: choosing her spending schedule before the more patient player chooses his. The game has a unique and fully polarized subgame perfect equilibrium, like the Nash equilibrium of the (fully) static game but with the impatient player using her first-mover advantage to shift spending toward the present. It illustrates that the strategic polarization observed in a static setting can be found in (at least something closer to) a true dynamic setting, but that the dynamics of a public good game can also introduce an asymmetry between the positions of more and less patient parties.

The dynamic game of the next section, which lies at the heart of this paper, will be used to explore this polarization and this asymmetry in more detail.

Uniqueness and measure-zero deviations

A household's optimal spending schedule, and the equilibria of the benchmark public good games, were described above as unique. Strictly speaking, since we will work in continuous time and assume that measure-zero deviations in spending rates do not affect payoffs, they are unique only up to measure-zero deviations.

Throughout the paper, we will take the standard approach of restricting our attention to right-continuous spending schedules—i.e. spending schedules that are right-continuous everywhere, and continuous everywhere except at a finite set of jump discontinuities—in both individual optimization decisions and strategic interactions. Since distinct right-continuous spending schedules differ from each other at a positive-measure set of times, a right-continuous spending schedule that satisfies some property uniquely up to measure-zero deviations also satisfies it uniquely under a right-continuity restriction. However, the "unique" optimal spending schedules and equilibria found in Propositions 1 and 3–5 are unique up to measure-zero deviations, not merely uniquely under the restriction of right-continuity. This is shown in the proofs of these propositions, in the relevant appendices.

The proofs of some later propositions use right-continuity, so the assumption of right-continuity is imposed throughout the body of the paper for consistency.

2.1 Private goods

Recall the familiar "cake-eating" problem of a household that is the sole provider of its own consumption over an infinite horizon. Denote the size of the household's budget at time $t = 0$ by B. Assume that flow utility u at time t is an isoelastic function,[6] with inverse elasticity of intertemporal substitution $\gamma > 0$, of the spending

[6] We assume isoelasticity until the generalization of Section 5 because time-separable preferences (2) are homothetic if and only if flow utility is isoelastic. Homotheticity is desirable here because Proposition 2, and the analogous "proportional willingness to pay" results of Section 4.3 that are at the heart of this paper, require it. Though similar results may obtain under weaker conditions, the inability to discuss proportional costs and benefits would come with some loss in clarity.

rate $x_t \geq 0$:

$$u(x_t) = \begin{cases} \frac{x_t^{1-\gamma}}{1-\gamma}, & \gamma \neq 1; \\ \ln(x_t), & \gamma = 1. \end{cases} \tag{1}$$

The household faces a constant instantaneous real interest rate r and a constant instantaneous time preference rate δ. The household's problem is then to choose a right-continuous schedule (i.e. spending schedule) $x = \{x_t\}_{t \geq 0}$ that maximizes

$$\int_0^\infty e^{-\delta t} u(x_t) dt \tag{2}$$

subject to the budget constraint

$$\int_0^\infty e^{-rt} x_t dt \leq B. \tag{3}$$

Let

$$\alpha \equiv \frac{r\gamma - r + \delta}{\gamma}. \tag{4}$$

Proposition 1. *Optimal private schedule*
Under budget constraint (3), if $\alpha > 0$, utility function (1)–(2) is uniquely maximized by schedule

$$x_t = B\alpha e^{(r-\alpha)t}, \quad t \geq 0. \tag{5}$$

If $\alpha \leq 0$, there is no optimal schedule.

Proof. Proofs may be found in many introductions to dynamic optimization (see e.g. Barro and Sala-i-Martin (2004), ch. 2.1), but for convenience, one may be found in Appendix A.1, along with an expression for the payoff to following the optimal schedule. □

Note that, since $\gamma > 0$ by assumption, the condition that $\alpha > 0$ is equivalent to the condition that

$$\delta > r(1 - \gamma). \tag{6}$$

If (6) holds, the optimal spending rate as a proportion of the budget at any time is constant and equal to α.

The optimal spending rate is sensitive to the discount rate: the lower δ is, the lower α is, and the more slowly it is optimal to spend. In fact, when (6) is violated, a lower proportional spending rate is always preferable to a higher one. Since a permanent spending rate of zero is of course worst of all, an optimal schedule does not exist. Note that (6) does hold whenever $\gamma > 1$, $r > 0$, and $\delta \geq 0$. That is, under the standard assumptions that $r > 0$ and $\gamma > 1$, an optimal schedule exists even under full patience.

As (5) reveals, the optimal private spending rate is continuous in δ, throughout the region in which an optimum exists. It is shown below that this continuity does not generally hold in the setting of public good provision.

Costs of over- or under-spending

Now imagine that a household with time preference rate δ is required to spend its budget as would be optimal according to some time preference rate $\tilde{\delta} \neq \delta$, fixing r and γ. Assume that δ and $\tilde{\delta}$ both satisfy (6), as above, but here further assume that

$$\begin{aligned} &\gamma \leq 1 \\ \text{or} \quad &\tilde{\delta} < \delta \frac{\gamma}{\gamma - 1} + r. \end{aligned} \tag{7}$$

If neither part of this condition is satisfied, the optimal schedule given $\tilde{\delta}$ sends spending to zero quickly enough that, over the infinite horizon, it produces infinite δ-discounted disutility. Note that $\tilde{\delta} < \delta + r$ is sufficient for the condition to hold.

Let $\tilde{\alpha}$ be defined as α is in (4), but with $\tilde{\delta}$ in place of δ. Then define

$$\eta \equiv \begin{cases} \left(\frac{\tilde{\alpha}+\delta-\tilde{\delta}}{\alpha}\right)^{\frac{1}{1-\gamma}}, & \gamma \neq 1; \\ e^{\frac{\tilde{\delta}}{\delta}-1}, & \gamma = 1. \end{cases} \tag{8}$$

η may be interpreted as a measure of the extent to which $\tilde{\delta}$ exceeds δ. Observe that $\eta > 1$ when $\tilde{\delta} > \delta$ and $\eta < 1$ when $\tilde{\delta} < \delta$.

Finally, let $w(B)$ denote the household's willingness to pay to avoid the spending rate requirement, as a proportion of its budget B.

Proposition 2. ***Bounded WTP for optimal private schedule***
$w(B) = 1 - \frac{\tilde{\alpha}}{\alpha\eta} < 1$, *independent of* B.

Proof. A proof may be found in Appendix A.2, along with an expression for the payoff to spending $\tilde{\delta}$-optimally. □

So the spending rate requirement—the requirement to spend at proportional rate $\tilde{\alpha}$ rather than α—lowers the household's utility by as much as a tax of proportion $1 - \frac{\tilde{\alpha}}{\alpha\eta}$ of its budget. As shown in Section 4.3, this cost can be greatly magnified in the setting of public good provision. In particular, as a patient player grows small, his proportional willingness to pay to avoid a spending rate minimum generally rises to 1.

2.2 Open-loop game: intertemporal polarization

Consider the following static game, concerning the allocation of spending on a public good across periods.

There are two players, H and L. Each player i chooses a piecewise right-continuous individual schedule $x^i \equiv \{x_t^i\}_{t\geq 0}$ subject to the budget constraint

$$\int_0^\infty e^{-rt} x_t^i dt \leq B^i.$$

A pair of individual schedules (x^H, x^L), called simply a "schedule", is denoted x. The sum of the players' schedules $x^H + x^L$ is called a "collective schedule" and denoted X.

H and L share an isoelastic flow utility function $u(\cdot)$, but now flow utility is a function of total flow spending by both parties: the good being spent on is a public good. The only difference between the players' preferences is that they discount at different rates. H discounts at the higher rate:

$$\delta^H > \delta^L > r(1-\gamma). \tag{9}$$

Thus i's payoff, as written as a function of the schedule, is

$$U^i(x) = \int_0^\infty e^{-\delta^i t} u\big(x_t^H + x_t^L\big) dt.$$

$U^i(\cdot)$ may also be written as a function of a collective schedule.

Let U denote a payoff profile (U^H, U^L). Define α^i as in (4), with δ^i in place of δ.

This game is the open-loop transformation of the dynamic public good game studied throughout the next section, so we will call it the "open-loop game".

Proposition 3. ***Existence and uniqueness of Nash equilibrium in the open-loop game***

The game above has a unique Nash equilibrium $x^{(o)}$*:*

$$x_t^{H(o)} = \begin{cases} B^H \alpha^H \, \frac{M^o}{M^o - 1} \, e^{(r-\alpha^H)t}, & t < t^o; \\ 0, & t \geq t^o, \end{cases} \tag{10}$$

$$x_t^{L(o)} = \begin{cases} 0, & t < t^o; \\ B^L \alpha^L \, M^{o\,\frac{\alpha^L}{\alpha^H}} \, e^{(r-\alpha^L)t}, & t \geq t^o, \end{cases} \tag{11}$$

where

$$t^o \equiv \ln(M^o)\big/\alpha^H, \tag{12}$$

$$M^o \equiv 1 + \frac{B^H \alpha^H}{B^L \alpha^L}. \tag{13}$$

Proof. See Appendix A.3. □

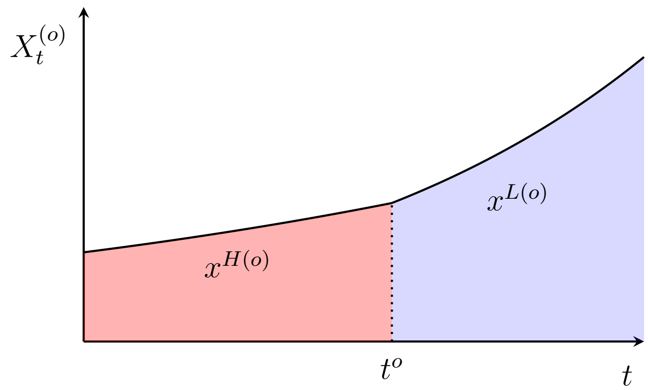

Fig. 1: Equilibrium schedule of the open-loop game

This game is essentially a special case of the static public good game analyzed by Bergstrom et al. (1986), but with a continuum of public goods: spending at each t, for $t \in [0, \infty)$. As in the analogous Bergstrom et al. case, each good here is provided by exactly one funder, and when a funder provides a good, she always also is the provider of all goods about which she cares relatively more. Since L cares relatively more than H about spending at t the later t is, there is a threshold time t^o such that H is the sole funder before t^o and L is the sole funder after. Unlike in the case of the private good, therefore, even a slight time preference difference motivates a big difference in behavior between the public good funders.

We can also see from (10) and (11) that i's spending growth rate here equals $r - \alpha^i$ across the period during which i spends. Since $r - \alpha^i = (r - \delta^i)/\gamma$, by (5) this is i's optimal spending growth rate: the growth rate i chooses when she is the only funder. If i's spending growth rate were not δ^i-optimal across t with $x_t^i > 0$, $x^{i(o)}$ would not be an equilibrium strategy; i would prefer marginal reallocations of funding across periods.

Finally, substituting (12) into (10) and (11), we see that the collective spending rate is continuous at t^o. That is, $\lim_{t \to t^{o-}} x_t^{H(o)} = x_{t^o}^{L(o)}$. If the spending rate rose discontinuously at t^o, L would do better to reallocate some spending from $t^o + \epsilon$ to $t^o - \epsilon$ for some sufficiently small $\epsilon > 0$. Likewise, if the spending rate fell discontinuously, H would do better to reallocate marginal spending forward. The relative budget sizes, the spending growth conditions, and the continuity condition pin down t^o and thus the equilibrium.

2.3 Stackelberg game: first-mover advantage

Finally, consider a two-period game, identical to the game above except that H is the "Stackelberg leader". That is, suppose that, in the first period, H sets a feasible spending schedule x^H, and in the second period, L observes x^H and sets a feasible spending schedule x^L in response. H's strategy set is thus the set of piecewise right-continuous spending schedules satisfying the budget constraint imposed by B^H, and

L's strategy set is the set of functions from feasible choices of x^H to feasible choices of x^L.

Define η as in (8), with δ^H in place of δ and δ^L in place of $\tilde{\delta}$. Since $\delta^L < \delta^H$, we have $\eta \in (0, 1)$.

Proposition 4. ***Existence and uniqueness of SPE in the Stackelberg game***

The game above has a unique subgame perfect equilibrium, which implements the schedule x^*:

$$x_t^{H*} = \begin{cases} B^H \alpha^H \, \frac{M^*}{M^*-1} \, e^{(r-\alpha^H)t}, & t < t^*; \\ 0, & t \geq t^*, \end{cases}$$

$$x_t^{L*} = \begin{cases} 0, & t < t^*; \\ B^L \alpha^L \, M^{* \, \frac{\alpha^L}{\alpha^H}} \, e^{(r-\alpha^L)t}, & t \geq t^*, \end{cases}$$

where

$$t^* \equiv \ln(M^*)/\alpha^H, \tag{14}$$

$$M^* \equiv 1 + \frac{B^H \alpha^H}{B^L \alpha^L} \, \eta.$$

Proof. See Appendix A.4. □

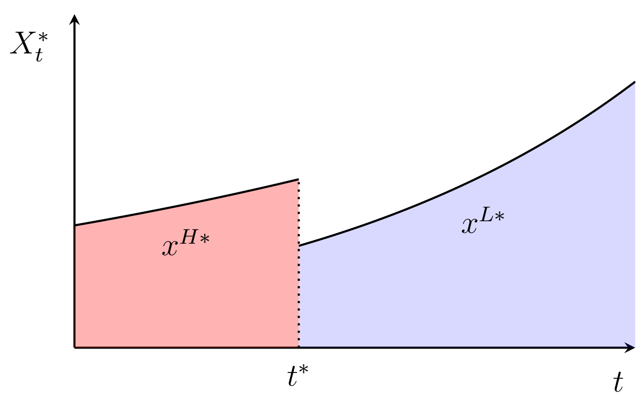

Fig. 2: Equilibrium schedule of the Stackelberg game

Since $\eta < 1$, $t^* < t^o$: the regime-switching time occurs earlier in the Stackelberg case than in the case where the players set their spending plans simultaneously. Furthermore, recall that in the open-loop (i.e. simultaneous-move) case, spending is continuous at t^o. Here, the spending rate falls discontinuously at t^*, as H allocates budget B^H over a shorter time interval and L allocates B^L over an infinite horizon beginning earlier.

$U^H(x^*) > U^H(x^{(o)})$. This follows immediately from the uniqueness of the SPE in the Stackelberg game, and from the fact that H can attain the open-loop schedule in it: if H chooses $x^H = x^{H*}$, L's best response is x^{L*} by Proposition 3. Conversely, $U^L(x^o) > U^L(x^*)$. This follows from the fact that X^* can be obtained by beginning with $X^{(o)}$, decreasing funding at times $t \geq t^*$, and increasing funding at times $t < t^*$. L disprefers any marginal reallocation of funding from times after t^* to times before t^* from the $X^{(o)}$ baseline, and because $u(\cdot)$ is concave, L only disprefers non-marginal reallocations the more strongly. In short, the Stackelberg game gives H a first-mover advantage.

3 Dynamic game

The open-loop and Stackelberg games suggest two ways in which time preference differences can shape public good provision. The first is by motivating what might be called *intertemporal polarization*. The second is by giving impatient parties a *first-mover advantage*. We will now see how these stylized results obtain in a more realistic dynamic setting, in which players can continuously observe each other's spending rates and update their own future spending plans. Equilibrium behavior in continuous time is defined precisely in Section 3.1, and the two implications above are detailed in Section 3.2, under an equilibrium refinement.

Unlike the benchmark settings, however, the dynamic setting admits a wide range of equilibria. This is discussed in Section 3.3.

3.1 Setup

Equilibrium behavior in the continuous-time dynamic setting will be defined as a limit of equilibrium behavior along a sequence of dynamic games with ever more frequent discrete-time monitoring, indexed by $n \geq 0$. A justification for this modeling assumption (and other complications that must be introduced) is given below, under "Equilibria and motivations".

As in the benchmark games, there are two players, H and L. Player i begins with budget $B^i > 0$ and faces a constant instantaneous real interest rate r. Player i's realized spending rate at t is denoted x^i_t, and i's payoff is

$$U^i(x) \equiv \int_0^\infty e^{-\delta^i t} u(x^H_t + x^L_t) dt, \tag{15}$$

where $u(\cdot)$ is isoelastic, as given by (1), and the time preference rates satisfy (9).

To summarize the section: game n partitions the non-negative real line into a (perhaps infinite) number of periods of positive length. At the beginning of each period, each player observes the history of play across previous periods and publicly chooses a time weakly before the end of the period at which she will announce her spending history since the start of the period. Each player then sets a spending plan until the first announcement. When i announces, $-i$ observes the history and sets a spending plan to the end of the period. If $-i$ has not yet announced in this period, then she also publicly plans an announcement at some time weakly before the end of the period, and i sets a spending plan until $-i$'s announcement.

Schedules

A schedule for i, denoted x^i, is again a spending rate for i at every time $t \geq 0$:

$$x^i \equiv \{x^i_t\}_{t \geq 0}.$$

Given an interval $I \subset [0, \infty)$ that is closed below and open above, a truncated schedule for i, denoted x^i_I, is a spending rate for i at every time $t \in I$:

$$x^i_I \equiv \{x^i_t\}_{t \in I}.$$

A truncated schedule for i with $I = [0, t)$, for finite t, is a partial schedule for i and may be denoted $x^i_{|t}$.

A schedule $x \equiv (x^H, x^L)$ is a pair of player schedules. A truncated or partial schedule, x_I or $x_{|t}$, is a pair of such player schedules over the same interval.

Given a schedule x, a collective schedule $X \equiv x^H + x^L$ is a total spending rate at every time $s \geq 0$. Truncated and partial collective schedules are defined and denoted analogously to such player schedules.

Given a truncated schedule x_I,

$$U^i(x_I) \equiv \int_I e^{-\delta^i (t - \min(I))} u\big(x^H_t + x^L_t\big) dt.$$

$U^i(\cdot)$ is also defined over collective [truncated] schedules. Payoff profiles $U(\cdot)$ are pairs of individual payoffs, as previously.

A partial schedule $x_{|t}$ is feasible if it is right-continuous and

$$\int_0^t e^{-rs} x^i_s ds \leq B^i \ \ \forall i.$$

The feasibility of a schedule is defined likewise.

Given a partial schedule $x_{|t}$,

$$B^i_{\underline{t}}(x_{|t}) \equiv \Big(B^i - \int_0^{\underline{t}} e^{-rs} x^i_s ds\Big) e^{r\underline{t}}$$

denotes i's implied budget at time $\underline{t} \leq t$. Given a schedule x, $B^i_{\underline{t}}(x)$ can be defined likewise, with no restriction on $\underline{t}$. For simplicity, $B^i(x_{|t}) \equiv B^i_t(x_{|t})$.

$B = B^H + B^L$ denotes the (initial) collective budget, and $b \equiv B^L/B$. Like the B^i, B and b can be written with time subscripts or as functions of a [partial] schedule.

Strategies

In each game, a set of times are called grid points. The set of grid points in game n is denoted $G^{(n)}$ and includes 0. The grid sequence $\mathcal{G} \equiv \{G^{(n)}\}_{n=0}^{\infty}$ is such that

- for all n, $G^{(n+1)} \supset G^{(n)}$ and $G^{(n)}$ is locally finite (so that $G^{(n)} \cap [0,T]$ is finite for all T); and
- $\cup_{n=0}^{\infty} G^{(n)}$ is dense in $\mathbb{R}_+$.

Given a time $t \geq 0$,

$$\tau^{(n)}(t) \equiv \max\left(G^{(n)} \cap [0,t]\right)$$

denotes the largest grid point in game n less than or equal to t, and $\tau^{(n)\prime}(t)$ denotes the smallest grid point in game n greater than t. Given $\tau \in G^{(n)}$, $\tau^{(n)\prime} \equiv \tau^{(n)\prime}(\tau)$ denotes the subsequent grid point in game n.

A period of game n is an interval $[\tau, \tau^{(n)\prime})$ with $\tau \in G^{(n)}$.

Henceforth, where doing so will come with no loss of clarity, we will fix n and drop the "(n)" superscripts.

An announcement ξ^i_τ is a time associated with a player i and a grid point τ with $\xi^i_\tau \in [\tau, \tau')$.

A node $h_{|t}$ is a history of play up to t: a feasible partial schedule across $[0,t)$, denoted $x(h_{|t})$, paired with a set of announcements $\{\xi^i_\tau(h_{|t})\}$ that includes exactly one entry-pair (one entry for each player) for each $\tau \in G \cap [0,t)$ and none for $\tau > t$, and such that (i) if $t \in G$ and $\xi^i_t(h_{|t})$ is defined for both i, then $\xi^H_t(h_{|t}) = \xi^L_t(h_{|t})$; and (ii) if $t \notin G$, $\xi^i_{\tau(t)}(h_{|t}) = t$ for some i.

If no announcement associated with some i, τ is contained in $\{\xi^i_\tau(h_{|t})\}$, we write $\xi^i_\tau(h_{|t}) = \emptyset$. Observe that every node $h_{|t}$ is of one of five types:

1. $t \in G$, $\xi^i_t(h_{|t}) = \emptyset$ for both i.
2. $t \in G$, $\xi^i_t(h_{|t}) > t$ for one or both i (and $= \emptyset$ otherwise).
3. $t \notin G$, $\xi^i_{\tau(t)}(h_{|t}) = \emptyset$ for one i (and $\xi^{-i}_{\tau(t)}(h_{|t}) = t$).
4. $t \notin G$, $\xi^i_{\tau(t)}(h_{|t}) > t$ for one i (and $\xi^{-i}_{\tau(t)}(h_{|t}) = t$).
5. $t \notin G$, $\xi^i_{\tau(t)}(h_{|t}) \leq t$ for both i (and $= t$ for one or both i).

The interpretation of these nodes is made clearer by the sequence of play outlined just below.

A node of type 1 or 3 is a pre-announcement node. A node of type 2, 4, or 5 is a post-announcement node. Given a post-announcement node $h_{|t}$,

$$\hat{\xi}(h_{|t}) \equiv \max_{i,\tau}\{\xi^i_\tau(h_{|t})\} \quad \text{if } h_{|t} \text{ is of type 2 or 4,}$$
$$\equiv \tau'(t) \quad \text{if } h_{|t} \text{ is of type 5}$$

denotes the time of the next node. We will sometimes distinguish pre- and post-announcement nodes occurring at the same time by denoting the former $h^-_{|t}$ and the latter $h^+_{|t}$.

The game begins with a trivial node of type 1 at $t = 0$.

Individual budgets $B^i(\cdot)$, the collective budget $B(\cdot)$, or L's budget share $b(\cdot)$ may be written as functions of nodes rather than of their associated partial schedules.

The set of nodes is denoted $\mathcal{H}$. We will say for simplicity that every element of $\mathcal{H}$ is a node for each player, though as we will see, some players take no substantive action at some nodes.

A strategy for i in game n, denoted σ^i, is a function that maps $h_{|t} \in \mathcal{H}$ to one of the following actions. The list below also defines how the subsequent node follows from the action profile $\sigma(h_{|t})$.

- If $h_{|t}$ is of type 1: $\sigma^i(h_{|t})$ is an announcement in $(t, t']$.
  The subsequent node occurs at t and is of type 2, and is $h_{|t}$ appended with $\xi^j_t = \sigma^j(h_{|t})$ if $\sigma^j(h_{|t}) \leq \sigma^{-j}(h_{|t})$ for each j.

- If $h_{|t}$ is of type 2: $\sigma^i(h_{|t})$ is a feasible spending plan to the time of the next node $\hat{\xi}(h_{|t})$.

  In general a feasible spending plan from node $h_{|t}$ to $\bar{t} > t$ is a truncated schedule for i, $x^i_{[t,\bar{t})}$, that is right-continuous and satisfies the budget constraint

  $$\int_t^{\bar{t}} e^{-r(s-t)} x^i_s ds \leq B^i(h_{|t}). \tag{16}$$

  If $\hat{\xi}(h_{|t}) = t'$, the subsequent node occurs at t' and is of type 1. If $\xi^H_t(h_{|t}) = \xi^L_t(h_{|t})(= \hat{\xi}(h_{|t})) < t'$, the subsequent node occurs at $\hat{\xi}(h_{|t})$ and is of type 5. Otherwise, the subsequent node occurs at $\hat{\xi}(h_{|t})$ and is of type 3. In all cases, the subsequent node is $h_{|t}$ with the associated partial schedule appended with the players' spending plans.

- If $h_{|t}$ is of type 3...

  a. and $\xi^i_{\tau(t)}(h_{|t}) = \emptyset$: $\sigma^i(h_{|t})$ is an announcement $\sigma^i(h_{|t}) \in (t, \tau'(t)]$.

b. and $\xi^i_{\tau(t)}(h_{|t}) = t$: $\sigma^i(h_{|t}) = \emptyset$.

Let j denote the player whose $\tau(t)$-announcement at $h_{|t}$ was undefined. The subsequent node occurs at $\sigma^j(h_{|t})$ and is of type 4, and is $h_{|t}$ appended with j's announcement.

- If $h_{|t}$ is of type 4: $\sigma^i(h_{|t})$ is a feasible spending plan to $\hat{\xi}(h_{|t})$.
  If $\hat{\xi}(h_{|t}) = \tau'(t)$, the subsequent node occurs at $\tau'(t)$ and is of type 1. Otherwise, the subsequent node occurs at $\hat{\xi}(h_{|t})$ and is of type 5. In either case, the subsequent node is $h_{|t}$ with the partial schedule appended with the players' spending plans.

- If $h_{|t}$ is of type 5: $\sigma^i(h_{|t})$ is a feasible spending plan to $\tau'(t)$.
  The subsequent node occurs at $\tau'(t)$ and is of type 1, and is $h_{|t}$ appended with the players' spending plans.

i's strategy set (in game n, though we will continue to suppress the superscript) is denoted Σ^i. A strategy profile (σ^H, σ^L) is denoted σ. The set of strategy profiles is denoted Σ. We will not introduce mixed strategies.

At $h_{|t} \in \mathcal{H}$, if players adopt strategy profile σ, the resulting schedule is denoted $x(h_{|t}, \sigma)$. For simplicity, $x(\sigma) \equiv x(\emptyset, \sigma)$ denotes the schedule generated by σ. The collective schedule resulting from σ following $h_{|t}$ is likewise denoted $X(h_{|t}, \sigma)$, etc. $h^-_{|s}(h_{|t}, \sigma)$ / $h^+_{|s}(h_{|t}, \sigma)$ is the pre-/post-announcement node implemented by σ at $s \geq t$ given $h_{|t}$, if such a node exists. (Note that it must if $s \in G$.)

Equilibria and motivations

We will take the natural equilibrium concept to be SPE. A strategy profile $\sigma^* \in \Sigma$ is an equilibrium (of game n) if

$$U^i\big(x_{[t,\infty)}(h_{|t}, \sigma^*)\big) \geq U^i\big(x_{[t,\infty)}(h_{|t}, (\sigma^i, \sigma^{*-i}))\big) \quad \forall i \; \forall h_{|t} \in \mathcal{H}. \tag{17}$$

Though including announcements makes almost no difference to the structure of the game when n is large, it simplifies our efforts to characterize its equilibria. Without announcements, in an equilibrium of game n in which H exhausts her budget, she typically does so at a grid point—an element of $G^{(n)}$—because she places some value on L immediately observing her budget-exhaustion and beginning to spend. So as we vary parameters of the game (e.g. as we vary the initial budgets at the node initiating a subgame, by exploring previous-period deviations), the time at which H chooses to exhaust her budget typically jumps between elements of $G^{(n)}$. Allowing the players to make announcements removes the additional value to H of spending down on a grid point, letting her equilibrium spend-down point vary continuously with the history (and the parameter values) of the game.

The complications induced by H's desire to spend down at a grid point could be addressed in various ways other than the approach taken here, but the alternatives explored introduce challenges of their own. Four of these are summarized below.

- If every time t were made a node, this would confront the difficulty that when nodes are dense in time, the sequence of play following a given node may not be well-defined as a function of the strategy profile. See e.g. Stinchcombe (2013), Example 4.2.1.

- If players were permitted to create mid-period nodes arbitrarily many times instead of just once, this would not allow us to solve for within-period behavior (or game-wide behavior, in the finite-horizon cases discussed below) by backward induction.

- The game could be defined so that, at $\xi^i \neq \xi^{-i}, \notin G$, i reveals his own recent spending to $-i$ and lets $-i$ change her plans, without learning $-i$'s recent spending or gaining the ability to change his own plans. This would render $x(h_{|t}, \sigma)$ undefined for some strategy profiles σ and nodes $h_{|t}$ with $t \notin G$. It would thus require a slightly more complex definition of equilibrium. After a natural re-definition, it would yield the same equilibrium spending behavior as obtains in the game defined above.

- The game could be defined so that players place mid-period nodes without announcing them in advance. However, the natural equilibrium concept is then not SPE but something imposing compatibility between actions and out-of-equilibrium beliefs.

  At $h_{|t}$, even if in equilibrium $-i$ places a node at $\xi^{-i} \in (t, \tau'(t))$, i must plan spending all the way to $\tau'(t)$ in the event that the expected announcement does not arrive. But SPE imposes no restriction on i's spending plans after ξ^{-i}, and unrealistic plans for x^i after ξ^{-i} (such as immediately spending i's entire budget) can in turn deter $-i$ from setting ξ^{-i} later. To rule out these "non-credible threats", we must then ensure that i's spending plans after ξ^{-i} are optimal for i given reasonable beliefs about $-i$'s behavior in the event that $-i$'s expected announcement does not arrive. This too appears to generate similar results but with significantly greater complexity.

Polarization

A polarized [truncated/partial] schedule is a [truncated/partial] schedule x such that, for some time $\tilde{t}$,

i. $x^H_t = 0$ for $t \geq \tilde{t}$ and

ii. $x^L_t = 0$ for $t < \tilde{t}$.

A polarized equilibrium (of game n) is an SPE σ^* such that, for all nodes $h_{|t}$, $x_{[t,\infty)}(h_{|t}, \sigma^*)$ is polarized.

Equilibrium schedules and payoffs

Schedule x is a [polarized] equilibrium schedule if there is a grid sequence and a sequence of [polarized] equilibria $\{\sigma^{(n)}\}$ along the corresponding game sequence such that

i. $x(\sigma^{(n)})$ converges pointwise to x almost everywhere and

ii. $U(x(\sigma^{(n)})) \to U(x)$

as $n \to \infty$.

Note that "polarized equilibrium schedules" are *schedules implemented by polarized equilibria*, rather than merely *equilibrium schedules that are polarized.* These are distinct because an equilibrium may implement a polarized schedule on the equilibrium path (i.e. from the initial node) but not from some off-path nodes.

The game sequences defined above admit many equilibrium schedules. To study the payoff implications of changing the parameters or structure of the games, therefore, it will often be useful to employ a refinement that guarantees uniqueness. As shown in the following subsection, uniqueness is guaranteed if we restrict our attention to equilibrium schedules that resemble the (unique) equilibrium schedules of finite-horizon transformations of the game. This is arguably the behavior it is most natural to take as a baseline, in the absence of coordination on some pattern of promised rewards and punishments.

Define T-horizon game n to be the discrete-time game n defined above, but with the ∞ at the top of the integrals in (15) and (17) replaced with a finite end-time $T > 0$. The definition of T-horizon equilibrium schedule follows immediately.

An equilibrium schedule x is a limit equilibrium schedule if there is a continuum of T-horizon equilibrium schedules $\{x^{[T]}\}$, for $T > 0$, such that $x^{[T]}$ converges pointwise almost everywhere to x as $T \to \infty$.

A payoff profile U is an equilibrium payoff if there is an equilibrium schedule x with $U(x) = U$.

Existence, uniqueness, and robustness

We impose relatively few restrictions on the grid sequence, and define an equilibrium schedule to be one compatible with any grid sequence, to ensure that results about the uniqueness of a given pattern of equilibrium behavior in continuous time are robust to alternative definitions of equilibrium behavior in continuous time.

The flexible definition of a grid sequence will also let us easily note when an existence result is robust, in that it is compatible with all grid sequences.

3.2 The polarized equilibrium

The open-loop and Stackelberg games have a unique equilibrium, and in each case the equilibrium schedule is polarized. The dynamic game has many equilibrium schedules, as noted in the next subsection. Nevertheless, there is a sense in which polarized behavior is most natural in the dynamic setting as well.

Proposition 5. ***Existence, uniqueness, and Stackelberg-equivalence of limit / polarized equilibrium schedule***
Defining x^* as the Stackelberg schedule of Proposition 4, in the dynamic setting of Section 3.1

a. *x^* is the unique limit equilibrium schedule.*
b. *x^* is the unique polarized equilibrium schedule.*
c. *x^* is an equilibrium schedule under any grid sequence.*

Proof. See Appendix A.5. □

Polarization, and the associated impatient first-mover advantage, are thus not only equilibrium-compatible in an infinite-horizon dynamic setting but uniquely compatible with a natural refinement. No matter how small the difference in discount rates, only polarized behavior is continuous with equilibrium behavior on an ever-lengthening finite horizon.

By contrast, suppose the players have the same discount rate δ, and let $X^{(\delta)}$ denote the δ-optimal collective schedule for the collective budget. Then, trivially, for any horizon (finite or infinite) and any set of grid points, any schedule x with $x^H + x^L = X^{(\delta)}$ is an equilibrium schedule.

3.3 Superior equilibrium payoffs

Proposition 6. ***Superior equilibrium payoffs***
Every feasible payoff profile which is Pareto-superior to $U(x^*)$ is an equilibrium payoff profile.

Proof. See Appendix A.6. □

As noted in Section 1.1, the standard intuition for efficiency results of this kind is that, given perfect monitoring in continuous time, deviating from an efficient equilibrium gives the deviator only an instantaneous benefit (say, by increasing his own consumption while free-riding on other players' public good contributions) but induces a positive-sized punishment (say, by decreasing subsequent contributions).

This intuition does not apply straightforwardly to this setting, where there is only a public good, and L's temptation to defect is a temptation to pay an instantaneous cost (by "underspending") for a long-term benefit.

Here, a proof of equilibrium-compatibility for every efficient feasible payoff profile Pareto-superior to $U(x^*)$ can be sketched as follows. A proof for the case of inefficient Pareto-superior payoff profiles is similar.

Given a collective budget B and an efficient collective schedule $X^{(a)}$ that is Pareto-superior to X^*, there is a range of values of b such that both players prefer $X^{(a)}$ to $X^*[b]$, the Stackelberg schedule that would result from b (fixing B). The range may be denoted $[\underline{b}_{X^{(e)}}, \overline{b}_{X^{(e)}}]$. Given the starting budget share for L b_0, and given efficient $X^{(a)}$ with $b_0 \in [\underline{b}_{X^{(a)}}, \overline{b}_{X^{(a)}}]$, can the parties be required to spend so that, for all t, we maintain $b_t \in [\underline{b}_{X^{(a)}_{[t,\infty)}}, \overline{b}_{X^{(a)}_{[t,\infty)}}]$? That is, can we fund $X^{(a)}$ while ensuring that b_t never falls low enough that H would prefer to switch to the Stackelberg schedule implemented given b_t, or rises high enough that L would? If so, there is an SPE in which $X^{(a)}$ is maintained, with deviations punished by reversions to the polarized equilibrium.

L's concern for the shape of the collective schedule in later periods is greater than H's (relative to their concerns for its shape in earlier periods). To be efficient, therefore, X^* must more closely resemble an impatient-optimal schedule early in time and a patient-optimal schedule later in time. So $\underline{b}_{X^{(a)}_{[t,\infty)}}$ and $\overline{b}_{X^{(a)}_{[t,\infty)}}$ increase in t.

An SPE can maintain the value of b around t by requiring the players to contribute to $X_t^{(a)}$ in proportion to their budgets around t. Less obviously, an SPE can induce b_t to rise more quickly than $\overline{b}_{X^{(a)}_{[t,\infty)}}$ around t by requiring H to contribute all of $X_t^{(a)}$. This can be shown by contradiction. Suppose H weakly prefers $X^{(a)}_{[t,\infty)}$ to the Stackelberg schedule that would result from budget-fraction b_t, but, after contributing $x_s^H = X_s^{(a)}$ for $s \in [t, \bar{t})$, disprefers $X^{(a)}_{[\bar{t},\infty)}$ to the Stackelberg schedule with budget-fraction $b_{\bar{t}}$. Then H prefers (i) $x_s^H = X_s^{(a)}, x_s^L = 0$ for $s \in [t, \bar{t})$, followed by a polarized schedule, to (ii) the collective schedule $X^{(a)}_{[t,\infty)}$. But (i) is a polarized truncated schedule, beginning at t, with the patient budget spent patient-optimally once L begins spending; and H's favorite truncated schedule in this class is the Stackelberg schedule. So H prefers the Stackelberg schedule with budget-fraction b_t to $X^{(a)}_{[t,\infty)}$. This contradicts our assumption.

4 Constraints

As noted in Section 2.2, a static public good game typically exhibits a unique Nash equilibrium, which is "polarized", in the sense that each project is funded entirely by the player(s) who most relatively value it. The prediction of polarization is relatively intuitive in a static context, as there is typically little to prevent polarization in a static public good game. A player is free not to contribute to any given project, and free to contribute as much as she would like. Section 3.2 offers a reason to expect a unique pattern of polarized behavior in a dynamic setting as well, when players have different discount rates and are free to spend as much or as little as they would like at any given time.

But intertemporal polarization may be constrained in two ways. First, a high-discount-rate player may not be able to spend down (the present value of) her budget quickly enough to achieve polarization, e.g. due to a borrowing constraint. Second, a low-discount-rate player may not be able to spend his budget slowly enough due to some sort of saving constraint, such as the legal disbursement minimum to which philanthropic foundations are often subject.

Sections 4.1 and 4.2 introduce these constraints respectively. Unsurprisingly, the constrained game can exhibit what might be called a constrained polarization, with one player bound by the spending minimum or maximum and the other spending more slowly or quickly than they would in isolation to offset the other player's putative over- or under-spending. In the constrained-polarized setting, however, sufficiently tight constraints of either kind are found to eliminate the first-mover advantage the impatient player enjoys under full polarization in the unconstrained setting.

Section 4.3 then explores the costs that these constraints impose on the players. An asymmetry is found: although spending minima counteract the first-mover advantage, spending minima in this setting can be arbitrarily more costly for the patient, in terms of proportional willingness to pay to remove them, than spending maxima can be for the impatient.

Comparative statics

Given a precisely defined spending constraint, let "constrained game n" be the constrained analog to game n. That is, let it be the game defined in Section 3.1 but with a strategy restriction imposed on the players: a condition that the spending plans assigned to each node must satisfy beyond the budget constraint (16) and right-continuity. The definition of an equilibrium schedule for the constrained game sequence then follows immediately.

To quantify the costs of a given constraint, we will identify a feasible schedule x that is in a natural sense as polarized as is feasible under the constraint, with H spending as quickly and/or L spending as slowly as is feasible (depending on the constraint(s) imposed). We will verify that x is an equilibrium schedule of the constrained game sequence. We will then compare the payoff profiles $U(x)$ and $U(x^*)$. Without picking particular equilibrium schedules in the constrained and unconstrained cases respectively, comparative statics quantifying the impacts of a given constraint are not well defined.

Our motivation for working with an infinite horizon is thus not the standard motivation to explore coordination through repeated interaction. Rather, we will work with an infinite horizon in order to most simply present the results of Section 4.3, which hold in the limit as the time horizon grows large.

Parameter restriction

Throughout this section, we will assume (in addition to parameter restriction (9)) that

$$\gamma \leq 1 \text{ or } \delta^H < \delta^L \frac{\gamma}{\gamma - 1} + r. \tag{18}$$

Note that (18) is (7) with δ^L, δ^H replacing δ, $\tilde{\delta}$ respectively. As in Proposition 2, the motivation for the restriction is to rule out cases in which δ^H-optimal spending yields infinite δ^L-discounted disutility.

4.1 Spending maxima / borrowing constraints

Given i's schedule x^i, let us refer to

$$x^i_t / B^i_t$$

as i's proportional spending rate at t. We will analyze spending maxima that take the form of a constant upper bound $\overline{\alpha}$ on the proportional spending rate that any player can exhibit at any time. Formally, we will restrict the games of Section 3 by requiring that, at each post-announcement node $h_{|t}$, i's spending plan $x^i_{[t,\xi(h_{|t}))}$ satisfy

$$x^i_s \leq B^i_s(x_{|s})\,\overline{\alpha} \tag{19}$$

in addition to the budget constraint (16) and right-continuity.

We will also assume that

$$\overline{\alpha} \in [\alpha^L, \alpha^H]. \tag{20}$$

We will maintain the existence of "announcements" to make the games studied here as comparable as possible to those of Section 3, though they could be eliminated without any impact on the results below.

Motivations

Though spending can of course be constrained in various ways, we will study a proportional spending cap (19) for two reasons.

First, our interest in spending maxima stems in part from the observation that polarization may be limited by the less patient player's borrowing constraints, and a cap on proportional spending is roughly equivalent to a borrowing constraint.

Suppose a player has a capital stock at t of $\tilde{B}_t^i$ and a stream of income—e.g. donor contributions—that grows at a constant rate $g < r$, so that inflows at s equal Ce^{gs} for some $C > 0$. The present value (at t) of her budget at t is then

$$B_t^i = \tilde{B}_t^i + \int_t^\infty Ce^{gs}e^{-r(s-t)}ds = \tilde{B}_t^i + \frac{Ce^{gt}}{r-g}.$$

If she cannot borrow against future income, her spending rate at t is unconstrained as long as $\tilde{B}_t^i > 0$. When $\tilde{B}_t^i = 0$ and her spending is constrained, however, she can spend at an absolute rate of at most Ce^{gt}, and so at a proportional rate of at most $r - g$. And if $\tilde{B}^i > 0$ but i would like to sustain a proportional spending rate greater than $r-g$, then unless she indefinitely refrains from doing so, she eventually exhausts her budget and confronts the proportional spending cap of $r - g$.

Second, upper bounds on proportional spending are precisely analogous to the legal disbursement minima—lower bounds on proportional spending—which are analyzed in Section 4.2 and which this paper is written in part to explore.

Our focus on the $\overline{\alpha} \in [\alpha^L, \alpha^H]$ case (20) also has two motivations.

First, this is the region which most straightforwardly illustrates the implications of constraining the impatient but not the patient from "polarizing". If $\overline{\alpha} < \alpha^L$, so that even L would prefer faster spending than is feasible, then there is little to analyze. The uniquely Pareto-dominant feasible schedule is the one in which $x_t^H/B_t^H = x_t^L/B_t^L = \overline{\alpha}\ \forall t$, as implementable e.g. with the simple SPE in which players spend as quickly as possible at every node. If $\overline{\alpha} > \alpha^H$, on the other hand, then the spending maximum only partially limits polarization by H, and adds kinks to the maximally-polarized equilibrium strategies that complicate the exposition to little benefit. (Recall from Proposition 1 that, in the absence of a more patient funder, H would choose to set $x_t^H/B_t^H = \alpha^H\ \forall t$. In any equilibrium-compatible polarized schedule in which H exhausts her budget at $\overline{t}^*$, $\lim_{t\to\overline{t}^{*-}} x_t^H/B_t^H = \infty$.)

Second, if the rate g at which the players' income streams grow equals the economic growth rate, and if a representative household is not borrowing- or saving-constrained, then the players' proportional spending cap $\overline{\alpha} = r - g$ equals the representative household's optimal proportional spending rate. When the public good under consideration is not a luxury or necessity good, $\overline{\alpha} \in [\alpha^L, \alpha^H]$ then amounts to the relatively natural assumption that its funders are not all strictly more or strictly less patient than the representative household.

Constrained polarization

Given a spending maximum $\overline{\alpha}$, a constrained-polarized schedule is a schedule x such that $x_t^H = B_t^H\overline{\alpha}$ for all t.

A constrained-polarized equilibrium (of game n) is an SPE σ^* such that, for all nodes $h_{|t}$, $x_{[t,\infty)}(h_{|t}, \sigma^*)$ is constrained-polarized.

A constrained-polarized equilibrium schedule is a schedule x such that there is a grid sequence and a sequence of constrained-polarized equilibria $\{\sigma^{(n)}\}$ across the corresponding game sequence such that $x(\sigma^{(n)})$ converges pointwise to x almost everywhere as $n \to \infty$.

In the absence of a spending maximum, the unique polarized (and unique limit) equilibrium schedule features $x_t^H > B_t^H \alpha^H \geq B_t^H \overline{\alpha}$ whenever $B_t^H > 0$. A constrained-polarized equilibrium schedule is therefore one in which H always chooses a spending rate as close as possible to that she chooses in a polarized equilibrium schedule in the absence of the constraint.

Define the "constrained open-loop game" to be the game in which players simultaneously choose individual schedules x^i, subject to (19), and receive payoffs (15).

Proposition 7. ***Existence and uniqueness of constrained-polarized equilibrium schedule given spending maximum***

Given spending maximum $\overline{\alpha} \in [\alpha^L, \alpha^H]$,

a. *There is a unique constrained-polarized equilibrium schedule* $\overline{x}$*:*

$$\overline{x}_t^H = B^H \overline{\alpha} e^{(r-\overline{\alpha})t},$$

$$\overline{x}_t^L = \begin{cases} 0, & t < \overline{t}^*; \\ \big(B^H e^{(r-\overline{\alpha})\overline{t}^*} + B^L e^{r\overline{t}^*}\big)\alpha^L e^{(r-\alpha^L)(t-\overline{t}^*)} - B^H \overline{\alpha} e^{(r-\overline{\alpha})t}, & t \geq \overline{t}^*, \end{cases}$$

where

$$\overline{t}^* \equiv \max\Big(0, \ln\Big(\frac{B^H}{B^L}\,\frac{\overline{\alpha}-\alpha^L}{\alpha^L}\Big)\Big/\overline{\alpha}\Big). \tag{21}$$

b. $\overline{x}$ *is an equilibrium schedule under any grid sequence.*

c. $\overline{x}$ *is an equilibrium of the constrained open-loop game.*

Proof. See Appendix A.7. □

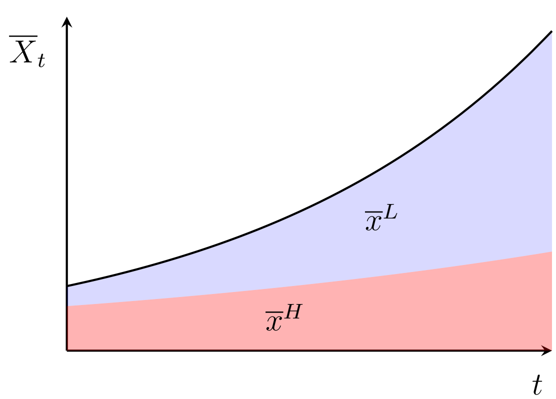

Fig. 3a: Constrained-polarized equilibrium schedule given spending maximum, b large

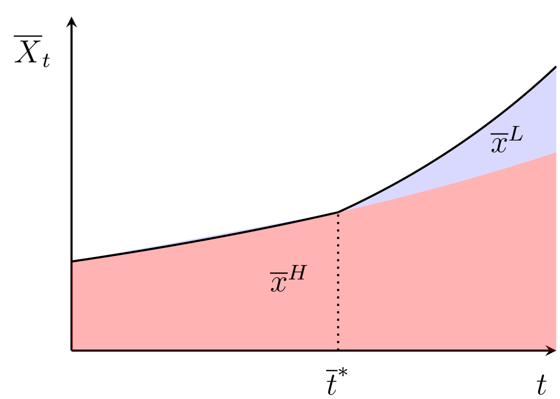

Fig. 3b: Constrained-polarized equilibrium schedule given spending maximum, b small

The spending maximum is binding for H, who spends at proportional rate $\overline{\alpha}$. L then allocates B^L patient-optimally, taking x^H to be independent of his own spending.

If

$$b > 1 - \alpha^L/\overline{\alpha}, \tag{22}$$

then H's fraction of the collective budget is small enough that at time zero her spending rate, $B^H\overline{\alpha}$, is less than the patient-optimal spending rate for the collective budget, $B\alpha^L$. L thus begins spending immediately, and achieves the patient-optimal collective schedule. If the inequality is reversed, H begins by overspending the collective budget, from L's perspective. L compensates by spending nothing until some $\overline{t}^* > 0$, and achieves the patient-optimal collective schedule subsequently.

A behavioral interpretation

Proposition 7 establishes that schedule $\overline{x}$ may obtain given that the players have the preferences from the dynamic setup, given in (15), and strategically interact to maximize them. Only the structure of the games, not the players' preferences, have been modified from Section 3.

Schedule $\overline{x}$ may also be given an alternate interpretation. Suppose that there are no spending constraints, but that H maximizes

$$\int_0^\infty e^{-\delta^H t} u(x_t^H) dt$$

instead of (15). That is, suppose that x^H is a private good for H, rather than X being a public good for both players. Then H chooses $x^H = \overline{x}^H$, in the $\overline{\alpha} = \alpha^H$ case, regardless of σ^L, by Proposition 1. If L still maximizes (15), his best response is $\overline{x}^L$. $\overline{x}$ (with $\overline{\alpha} = \alpha^H$) is thus the schedule that obtains if, in the terminology of Andreoni (1990), H is a warm-glow funder and L is an altruistic funder.

Scenarios in which a less patient player is effectively warm-glow and a more patient player is effectively altruistic may be relevant in practice not only because some funders are in fact warm-glow, but also because a less patient funder may have uncertainty about whether future funding from a patient funder will ever materialize. As noted in the introduction, slow-spending philanthropists sometimes face this very skepticism. The model of Section 3 is then one in which H perfectly anticipates future spending by L, and the model of this section explores the other end of the spectrum, with H expecting no future spending by L at all—at least until he is wealthy enough to implement his favorite allocation of the collective budget.

4.2 Spending minima / saving constraints

We will now explore the implications of a spending minimum $\underline{\alpha}$, in isolation and then in combination with a spending maximum $\overline{\alpha} \geq \underline{\alpha}$. Unlike spending maxima, spending minima are often imposed by law. In the United States, for instance, charitable foundations must disburse at least 5% of their assets each year.

As with the spending maxima of the previous subsection, we will restrict our attention to the $\underline{\alpha} \in [\alpha^L, \alpha^H]$ case. This is again because it is arguably both the most economically interesting case and the most realistic, for roughly the same reasons.

As noted in Section 4.1, borrowing constraints can constrain philanthropic spending on some public good from growing more slowly than the economic growth rate g, which in turn is the optimal spending growth rate from the perspective of a representative household if the public good is not a luxury or a necessity. Disbursement requirements are also sometimes motivated by a desire to prevent foundations from growing quickly enough to accumulate outsize influence.[7] On either grounds, we might expect to find a legal spending minimum of $r-g$, so that philanthropic spending by all parties grows at a rate no higher than g. In particular, the standard rules of thumb that the real rate of return on capital is 7% per year and the real economic growth rate is 2% per year might be expected to motivate a 5% disbursement minimum, as observed. And again, unless L and H are both unusually patient or impatient, a minimum spending rate of $r-g$ will lie in $[\alpha^L, \alpha^H]$.

Given a spending minimum $\underline{\alpha}$ [and maximum $\overline{\alpha}$], a constrained-polarized schedule is a schedule x such that $x_t^L = B_t^L \underline{\alpha}$ [and $x_t^H = B_t^H \overline{\alpha}$] for all t. Constrained-polarized equilibria and constrained-polarized equilibrium schedules are then defined and motivated as in Section 4.1.

Proposition 8. ***Existence and uniqueness of constrained-polarized equilibrium schedule given spending minimum***

Given spending minimum $\underline{\alpha} \in [\alpha^L, \alpha^H]$,

a. There is a unique constrained-polarized equilibrium schedule $\underline{x}$:

$$\underline{x}_t^H = \begin{cases} B^L \alpha^L \left(e^{(\alpha^H - \alpha^L)\underline{t}^* + (r-\alpha^H)t} - e^{(r-\alpha^L)t} \right), & t < \underline{t}^*; \\ 0, & t \geq \underline{t}^*, \end{cases}$$

$$\underline{x}_t^L = B^L \underline{\alpha} e^{(r-\underline{\alpha})t},$$

where $\underline{t}^*$ uniquely satisfies

$$\alpha^H / b = \underline{\alpha} e^{(\alpha^H - \underline{\alpha})\underline{t}^*} + (\alpha^H - \underline{\alpha}) e^{-\underline{\alpha t}^*}$$

(or equals ∞ if $b = 0$, or if $\underline{\alpha} = \alpha^H$).

b. $\underline{x}$ is an equilibrium schedule under any grid sequence.

c. $\underline{x}$ is an equilibrium of the constrained open-loop game.

[7] In the extreme, see fears that trusts with plans to invest for too long before disbursing would "shatter the nation's financial structure", re-"fashion [the] economy", and ultimately "amount to all the total value of the world" (O'Kane, 1961).

Proof. See Appendix A.8. □

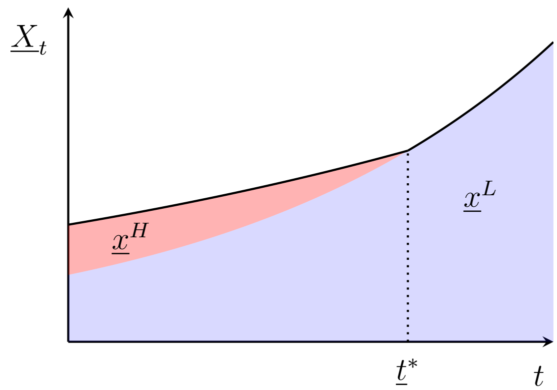

Fig. 4: Constrained-polarized equilibrium schedule given spending minimum

Proposition 9. ***Existence and uniqueness of constrained-polarized limit equilibrium schedule given spending minimum and maximum***

Given spending minimum $\underline{\alpha}$ and spending maximum $\overline{\alpha}$ with $\alpha^L \leq \underline{\alpha} \leq \overline{\alpha} \leq \alpha^H$,

a. There is a unique limit equilibrium schedule $\overline{\underline{x}}$:

$$\overline{\underline{x}}_t^H = B^H \overline{\alpha} e^{(r-\overline{\alpha})t},$$

$$\overline{\underline{x}}_t^L = B^L \underline{\alpha} e^{(r-\underline{\alpha})t}.$$

b. $\overline{\underline{x}}$ is an equilibrium schedule under any grid sequence.
c. $\overline{\underline{x}}$ is an equilibrium of the constrained open-loop game.

Proof. See Appendix A.9. □

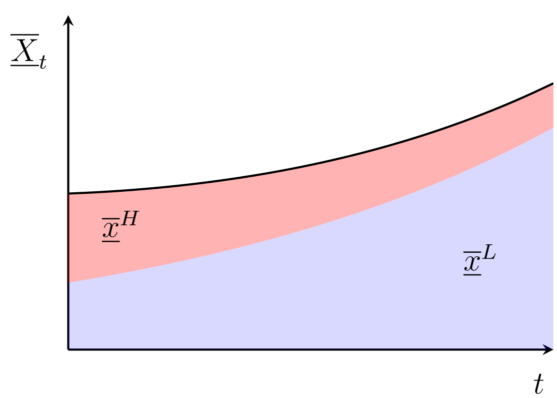

Fig. 5: Constrained-polarized equilibrium schedule given spending minimum and maximum

4.3 The costs of constraints

Despite the symmetry of the spending minima and maxima we have introduced, and the fact that polarized behavior in the unconstrained case gives the impatient player a first-mover advantage, it is the patient player especially who can find a spending constraint costly.

To express this result succinctly, fix r, γ, δ^H, and δ^L satisfying (9) and (18). Then let

$$x[B^H, B^L, \underline{\alpha}, \overline{\alpha}]$$

denote the [constrained-]polarized equilibrium schedule given positive budgets B^L and B^H and a (potentially vacuous) spending minimum $\underline{\alpha}$ and maximum $\overline{\alpha}$ satisfying

$$\underline{\alpha} \in 0 \cup [\alpha^L, \alpha^H], \quad \overline{\alpha} \in [\alpha^L, \alpha^H] \cup \infty, \quad \overline{\alpha} \geq \underline{\alpha}. \tag{23}$$

Finally, given (23), let w^i denote i's proportional willingness to pay to fully relax the relevant spending constraint:

$$w^H(b, \underline{\alpha}, \overline{\alpha}) \equiv w : U^H(x[1-b, b, \underline{\alpha}, \overline{\alpha}]) = U^H(x[(1-w)(1-b), b, \underline{\alpha}, \infty]),$$
$$w^L(b, \underline{\alpha}, \overline{\alpha}) \equiv w : U^L(x[1-b, b, \underline{\alpha}, \overline{\alpha}]) = U^L(x[1-b, (1-w)b, 0, \overline{\alpha}]).$$

The willingness to pay function $w^i(\cdot)$ is given as a function of L's budget share $b \in (0,1)$, rather than of B^H and B^L, because holding b fixed, x^*, $\overline{x}$, $\underline{x}$, and $\underline{\overline{x}}$ are linear in B. Also, and relatedly, $U(\cdot)$ is homothetic: given any two schedules to which a player assigns the same payoff, she also assigns the same payoffs to corresponding schedules in which spending at each period is multiplied by a constant $B > 0$. Without loss of generality, therefore, when computing proportional willingness to pay, we can restrict our attention to the $B = 1$ case.

Proposition 10. ***Asymmetric costs of spending constraints***

Given spending constraints $\underline{\alpha}, \overline{\alpha}$ satisfying (23),

a. $\sup_b \; w^H(b, \underline{\alpha}, \overline{\alpha}) < 1$.

b. $\lim_{b \to 0} \; w^L(b, \underline{\alpha}, \overline{\alpha}) = 1$ *if* $\underline{\alpha} \geq \alpha^L$ *and* $\overline{\alpha} > \alpha^L$.

Proof. See Appendix A.10. □

In other words, when the more patient player controls a small proportion of the funding in some domain, a spending minimum is approximately as costly to him as a total expropriation, at least if spending is maximally polarized—even a spending minimum of α^L. This is true whether or not there is also a spending maximum (as long as the spending maximum is not also α^L). A spending maximum is not approximately as costly to the less patient player as a total expropriation, however, and this too is true whether or not there is also a spending minimum.

Recall from Proposition 2 that no result analogous to Proposition 10b obtains in the case of private good provision. This result highlights a welfare-relevant respect in which time preference heterogeneity is especially important in the context of public good provision.

An intuition for the asymmetry in the cost of a spending constraint is clearest when the baseline schedule is constrained in both directions.

Removing a spending maximum does not change x^L in the constrained-polarized equilibrium schedule. When B^H/B is small—i.e. when $b \approx 1$—it simply concentrates H's spending around $t = 0$.

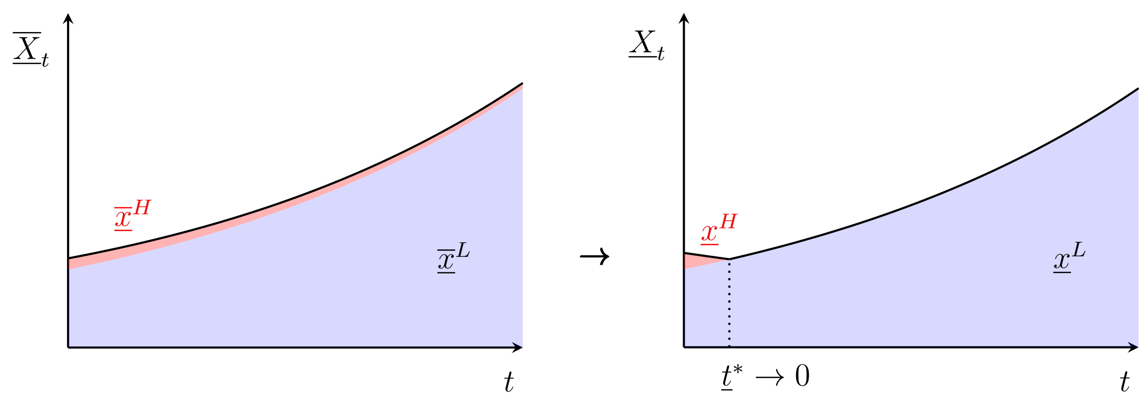

Fig. 6a: Limit equilibrium schedule given binding spending minimum and maximum, $b \to 1$

Fig. 6b: Limit equilibrium schedule given binding spending minimum alone, $b \to 1$

Under both constraints, when $b \approx 1$, the utility that H's budget offers her, above the baseline $U^H(x^L)$ provided by L's spending, is roughly a discounted average of $e^{(r-\delta^H)t}u'(x_t^L)$ across $t \geq 0$ per unit of B^H. When the spending maximum is lifted, the utility that H's budget offers her above the baseline is roughly $u'(x_0^L)$ per unit of B^H. This is a bounded improvement.

With and without the spending minimum, the growth rate of x^H equals $r - \overline{\alpha}$. Given $\overline{\alpha} > \alpha^L$, is strictly less than the δ^L-optimal spending growth rate of $r - \alpha^L$. So $e^{(r-\delta^L)t}u'(x_t^H)$, the δ^L-discounted marginal utility of an increase in resources allocated to t, grows unboundedly—indeed, exponentially—with t.

Under both constraints, regardless of b, L spreads his spending over time in such a way that most of the present value of his budget is spent before some sufficiently large T. This bounds the additional utility that B^L offers L above the $U^L(x^H)$ baseline.

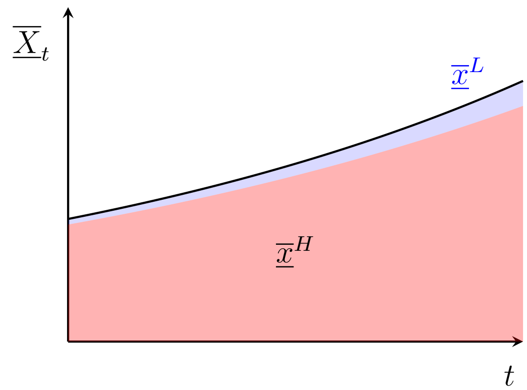

Fig. 7a: Limit equilibrium schedule given binding spending minimum and maximum, $b \to 0$

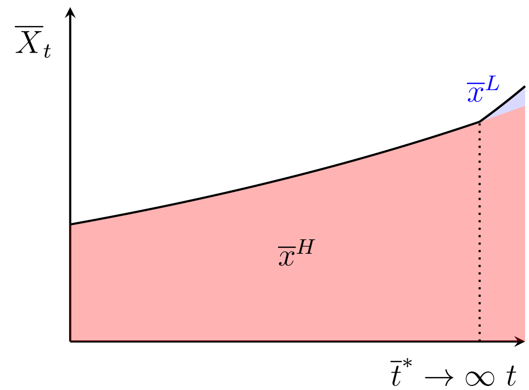

Fig. 7b: Limit equilibrium schedule given binding spending maximum alone, $b \to 0$

When the spending minimum is lifted, however, the time $\overline{t}^*$ at which L begins spending grows without bound as $b \to 0$. The utility that L's budget offers him above the baseline thus rises to infinity per unit of B^L as $b \to 0$.

Constraints and efficiency

Though efficiency is not the primary focus of this paper, we close this section by noting a way in which constraints on public good spending rates, despite their potentially large costs for one player, can bring about efficiency.

Recall from Proposition 6 that, in the absence of constraints, any efficient payoff Pareto-superior to $U(x^*)$ is an equilibrium payoff. There are also inefficient equilibrium schedules, however, including the polarized equilibrium schedule.

Under a one-sided constraint, the constrained-polarized equilibrium schedule is typically also inefficient. This can be seen from the fact that it typically exhibits a kink in collective spending which both players would prefer to smooth. But if the constraint is tight enough that it implements one player's favorite allocation of the collective budget, the associated schedule is efficient.

Under a two-sided constraint, the constrained-polarized equilibrium schedule features no kink. Instead, (given $\overline{\alpha} > \underline{\alpha}$), the collective proportional spending rate falls smoothly from $b\underline{\alpha} + (1-b)\overline{\alpha}$ to $\underline{\alpha}$ as L gradually comes to possess a larger share of the collective remaining budget. Under certain parameter conditions, this produces an efficient schedule that is not either player's most preferred schedule.

Proposition 11. ***Constraints and efficiency***

Given spending constraints $\underline{\alpha}, \overline{\alpha}$ *satisfying (23),* $x[B^H, B^L, \underline{\alpha}, \overline{\alpha}]$ *is efficient iff*

i. $\underline{\alpha} = \alpha^H$*;*

ii. $\overline{\alpha} = \alpha^L$*;*

iii. $\underline{\alpha} = 0$, $\overline{\alpha} \geq \alpha^L$, *and* $b > 1 - \alpha^L/\alpha^H$*; or*

iv. $\underline{\alpha} = \alpha^L$, $\overline{\alpha} = \alpha^H$, *and* $\gamma = 1$,

maximizing U^H *given (i),* U^L *given (ii)-(iii), and* $(1-b)U^H + bU^L$ *given (iv).*

Proof. See Appendix A.11. □

5 Delayed benefits

We have assumed so far that the flow utility the players enjoy at each time depends only on collective spending at that time. This assumption most directly describes the case in which the public good "X" is a perishable consumption good. Often, however, public good expenditures generate benefits after the expenditures occur. Spending on infrastructure, education, research, and environmental preservation is all typically intended to generate a stream of benefits lasting far beyond the day the money is spent. One may therefore wonder whether, in these domains, the basic logic of intertemporal polarization explored in earlier sections is maintained. We will now show that, under broad conditions, it is.

The notion that a more patient player would want to cut such expenditures when a less patient player would want to increase them may be counterintuitive, since calls to increase such expenditures in a political setting are typically made on the basis of their future benefits. But in such a setting, the options at hand are typically (i) to spend on a project with delayed benefits or (ii) to leave the required resources untaxed, in which case at least some of them will be consumed in the short term. By contrast, the setting studied here concerns private funders, such as foundations or charitable trusts, all of whose budgets will be spent on public goods at some time. The options at hand are therefore (i) to spend on a project today or (ii) to invest the entirety of the required resources to a future date and fund more projects then. If an education funder is indifferent between building one school today and two schools in 20 years, it is perhaps not so counterintuitive to suppose that a more patient education funder will prefer the latter.

To formalize this point simply, we will work in discrete time. Let X denote a discrete-time collective schedule $\{X_t\}_{t \in \mathbb{N}}$. Suppose that each player's utility at t equals $u_t(X)$: an arbitrary time-specific function of collective spending X_s for each $s \in \mathbb{N}$. Player i has a sequence of time preference rates δ^i_t, with $\delta^H_t > \delta^L_t \; \forall t$, so that i's payoff is

$$U^i(X) = \sum_{t=0}^{\infty} e^{-\sum_{q=1}^{t} \delta^i_q} u_t(X). \tag{24}$$

Note that we do not require u_t to be increasing in each X_s: u_t may be constant or decreasing in X_s, locally or globally, for some or all s. In particular, we do not require u_t to be constant in X_s for $s < t$: the plan to invest a unit of resources until t may generate costs or benefits before t, e.g. if the invested resources support environmentally destructive or socially impactful firms.

Because $u_t(\cdot)$ is time-varying, this framework is flexible enough to incorporate the effects of any funders outside the two whose spending constitutes X. For illustration, suppose flow utility at t is the logarithm of spending at t by H, L, and a third party. If the third party exogenously spends one unit at t, then $u_t(X) = \log(X_t + 1)$. If the third party is fully crowded out of spending at t as H and L increase their own spending at t, then $u_t(X) = \log(\max(1, X_t))$.

Like discount rates, we do not require that interest rates be constant or positive. Instead, let $R_0 = 1$, and let $R_t > 0$ denote the cumulative returns to investing from 0 to t.

$\{u_t(\cdot)\}$ satisfies <u>single crossing from X</u> if the function-sequence is differentiable at X and for each pair of times $\underline{s} < \overline{s}$,

$$\exists \tilde{t} \in \{0, 1, ..., \infty\} \; : \; \frac{\partial u_t(X)}{\partial X_{\underline{s}}} \geq \frac{R_{\underline{s}}}{R_{\overline{s}}} \frac{\partial u_t(X)}{\partial X_{\overline{s}}} \;\; \forall t < \tilde{t}, \tag{25}$$

$$\frac{\partial u_t(X)}{\partial X_{\underline{s}}} \leq \frac{R_{\underline{s}}}{R_{\overline{s}}} \frac{\partial u_t(X)}{\partial X_{\overline{s}}} \;\; \forall t > \tilde{t}; \text{ and}$$

$$\exists t \; : \; \frac{\partial u_t(X)}{\partial X_{\underline{s}}} \neq \frac{R_{\underline{s}}}{R_{\overline{s}}} \frac{\partial u_t(X)}{\partial X_{\overline{s}}}. \tag{26}$$

That is, single crossing obtains from X if a marginal resource allocation to some time $\underline{s}$ produces a stream of undiscounted net benefits that is not equal to the benefit-stream produced by a marginal resource allocation to $\overline{s} > \underline{s}$, but lies weakly above it, lies weakly below it, or crosses it once from above.

The proposition below establishes that single crossing is closely associated with intertemporal polarization. First we will argue for the plausibility of the property across familiar domains in which spending brings delayed benefits.

Suppose $\{u_t(\cdot)\}$ and X are such that marginal education spending at s produces a stream of benefits that begins at C_s per unit of expenditure and then grows exponentially at rate g.[8] Then for any pair $\underline{s} < \overline{s}$, the stream of benefits to allocating marginal resources for education expenditure at $\underline{s}$ lies weakly above that for $\overline{s}$ if $C_{\overline{s}} \leq C_s \, R_s / R_{\overline{s}}$, and the streams cross once (at $\overline{s}$) if $C_{\overline{s}} > C_{\underline{s}} \, R_{\underline{s}} / R_{\overline{s}}$. In either case, single-crossing from X obtains.

Or consider the case of avoiding a catastrophe severe enough to cause human extinction. Because the effects of such a catastrophe would be permanent, the effort to avoid one is sometimes used as the paradigmatic example of a domain in which the

[8] This would roughly be implied by the Lucas (1988) model of human capital accumulation, for instance.

patient desire more spending than the impatient. Suppose that, given X, the flow utility associated with the existence of humanity at t is $v_t > 0$ and the probability of survival to t is p_t, with $p_0 = 1$ and $p_{(\cdot)}$ non-increasing. Then (expected) utility at t is $u_t(X) = p_t v_t$. Suppose also that marginal spending at s lowers the probability of an existential catastrophe at s by C_s per unit spent, given survival to s (resources allocated to s having no effect absent survival to s). Then the stream of expected benefits generated by allocating marginal resources to s is $\{R_s C_s p_t v_t\}_{t \geq s}$. As in the education case, for any pair $\underline{s} < \overline{s}$, the benefit-stream to allocating marginal resources to $\underline{s}$ lies weakly above that for $\overline{s}$ if $C_{\overline{s}} \leq C_s \, R_s / R_{\overline{s}}$, and the streams cross once, at $\overline{s}$, if $C_{\overline{s}} > C_{\underline{s}} \, R_{\underline{s}} / R_{\overline{s}}$.

As in earlier sections, suppose the players begin with budgets $\{B^i\} > 0$, with $B \equiv B^H + B^L$. A collective schedule X is <u>feasible</u> if $\sum_{t=0}^{\infty} X_t / R_t \leq B$, and an individual schedule for i, $x^i \equiv \{x^i_t\}_{t \in \mathbb{N}}$, is feasible if $\sum_{t=0}^{\infty} x^i_t / R_t \leq B^i$.

The <u>generalized open-loop game</u> is the game in which the players simultaneously choose feasible individual schedules and earn payoffs (24).

Proposition 12. ***Polarization and single crossing***

If $\{u_t(\cdot)\}$ satisfies single crossing

a. *from X, then given $\underline{s} < \overline{s}$,*

$$\frac{\partial U^H(X)}{\partial X_{\overline{s}}} \geq \frac{R_{\underline{s}}}{R_{\overline{s}}} \frac{\partial U^H(X)}{\partial X_{\underline{s}}} \implies \frac{\partial U^L(X)}{\partial X_{\overline{s}}} > \frac{R_{\underline{s}}}{R_{\overline{s}}} \frac{\partial U^L(X)}{\partial X_{\overline{s}}},$$

$$\frac{\partial U^L(X)}{\partial X_{\overline{s}}} \leq \frac{R_{\underline{s}}}{R_{\overline{s}}} \frac{\partial U^L(X)}{\partial X_{\underline{s}}} \implies \frac{\partial U^H(X)}{\partial X_{\overline{s}}} < \frac{R_{\underline{s}}}{R_{\overline{s}}} \frac{\partial U^H(X)}{\partial X_{\underline{s}}}.$$

b. *from all feasible X, then in any equilibrium x of the generalized open-loop game,*

$$\exists \tilde{s} : \; x^H_s = 0 \;\; \forall s > \tilde{s}, \;\; x^L_s = 0 \;\; \forall s < \tilde{s}.$$

Proof. See Appendix A.12. □

It appears that an analysis of dynamic behavior in this generalized setting would be very complex, and it will not be pursued here. Nevertheless, Proposition 12 and the discussion above suggests that, in isolation, the fact that public good spending often produces delayed benefits does not typically alter the basic logic of intertemporal polarization. Patience may motivate postponing one's philanthropy.

6 Conclusion

The economic implications of time preference heterogeneity have been extensively explored in a variety of domains, including social discounting, optimal taxation, and dynamic bargaining. They have to date, however, largely been overlooked in literature on the private provision of public goods. The results presented here begin to fill this gap, and illustrate the importance of building time preference heterogeneity into models of dynamic public good provision going forward.

In dynamic public good provision contexts, time preference heterogeneity appears both even more widespread and even more important than in other contexts.

For the most important dynamic public good games in practice, there is unusually clear empirical and theoretical evidence that their players do not employ common discounting, as noted in Section 1.1.

The results of this paper illustrate that time preference heterogeneity is not just particularly widespread in dynamic public good provision settings but also particularly important. As a comparison between the cases of Section 2.1 and Section 3 illustrates, equilibrium behavior in dynamic public good games with versus without time preference heterogeneity can differ dramatically: a player might spend at a positive rate at time zero under common discounting but, given a time preference rate only slightly below that of his co-funder, spend nothing for a long while. And even under arbitrarily small time preference differences, spending constraints can have arbitrarily large proportional welfare implications for small patient philanthropists in a large impatient world, in the sense explored in Section 4.3 (though not the reverse). Spending constraints typically have no such extreme implications in the private good context.

The pervasiveness and importance of time preference heterogeneity in public good provision contexts has at least two broad classes of policy implications.

First, it affects the structure of self-enforcing agreements for the provision of public goods among multiple parties who cannot contract, such as national governments. Such agreements can be efficient, as shown in Section 3.3, at least in continuous time and with perfect monitoring. If agreements are designed without accounting for the parties' different discount rates, however, they will generally not be efficient, and may fail to be self-enforcing as intended. The United States's 2017 withdrawal from the Paris Agreement on climate change, for instance, coincided with an explicit impatient shift in Office of Management and Budget policies on social discounting, as requested by the newly elected president.[9] If Americans are simply less patient than their international counterparts, this will in the long run affect democratically-set discounting policy, and punishments (here taking the form of increases in collective

[9] See e.g. Li and Pizer (2021).

emissions) sufficient to deter defections by many countries may be insufficient to deter defections by Americans.

Second, an understanding of the importance of time preference heterogeneity for public good provision might affect the policy conversation around disbursement requirements for philanthropic foundations, trusts, and DAFs. Voters and policy-makers might look more charitably on slow- or non-disbursing charitable investment vehicles once a lack of disbursement is understood not as proof of a tax-avoidance scheme but as, at least possibly, a logical consequence of patient philanthropic planning. Estimates of a patient philanthropist's high ideal willingness to pay to avoid disbursement requirements may also motivate patient philanthropists themselves to fight disbursement requirements more vigorously.

The lessons emphasized here are demonstrated rigorously in the context in which there is a single good and each player's flow utility is an isoelastic function of total flow spending on it. The robustness of the lessons is suggested by the result of Section 5 that intertemporal polarization, at least, is maintained when the framework is extended to allow for much more general preferences.

These results and discussions however only begin to cover the space of possible implications of time preference heterogeneity for public good provision. We have attempted to isolate the implications of time preference heterogeneity by focusing on a model with only two players and a single public good; common knowledge of preferences and of the value of the good; perfect monitoring; and a risk-free AK investment environment. Even in this highly restricted setting, the analysis is incomplete without further work on equilibrium characterization and selection. To quantify the benefits of coordination, for instance, it may be valuable to explore equilibrium selection among efficient SPEs, perhaps using tools from the considerable existing literature on dynamic bargaining under time preference heterogeneity.

More work, both theoretical and empirical, would also be necessary to understand the implications of time preference heterogeneity in real-world public good provision problems which do not conform to the restrictions listed above. For instance, governments and philanthropists generally face a variety of public good domains, which produce benefits on different time horizons. Time preference heterogeneity might then be expected to generate polarization both across domains and across time, with patient parties funding an ever larger share of (ever shorter-horizon) domains as they grow to constitute ever larger proportions of the total philanthropic market. Also, given the possibility of risky investments, since the range of projects covered by each player-type will be endogenous to the players' budget shares, a public good funder's own time preferences and those of other philanthropists presumably have implications for the funder's own financial risk-tolerance. Exploring the implications of time preference heterogeneity across the dimensions of both time and domain further would appear to be a valuable topic for future research.

Even the simple model explored here confronts a host of complexities, in part because many useful results from the literature on repeated games (like algorithms to characterize the set of equilibrium payoffs) cannot be used. Extensions along the lines above would doubtless face even more difficulties. Given the pervasive importance of time preference heterogeneity to dynamic public good provision, however, such efforts appear to be worthwhile.

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A Proofs

A.1 Proof of Proposition 1 and corresponding payoff

Proof of Proposition 1

Given a schedule x, let

$$y_t \equiv e^{-rt} x_t$$

denote the density of resources allocated at time 0 for investment until, followed by spending at, t. The schedule y of present-value spending rates will be called an "allocation", and y_t a "flow allocation". Let

$$v(y_t) \equiv e^{-\delta t} u(e^{rt} y_t)$$

denote the discounted flow utility at t from flow allocation y_t.

Because marginal utility in spending is always positive, y is optimal only if it exhausts the budget.

A budget-exhausting allocation y maximizes utility iff, given y, the marginal flow utility of allocating to each t equals some constant λ almost everywhere:

$$\begin{aligned} v'(y_t) &= \frac{\partial}{\partial y_t}\Big[e^{-\delta t}\frac{(e^{rt}y_t)^{1-\gamma}-1}{1-\gamma}\Big] = \lambda \ \forall\forall t, \quad \gamma \neq 1; \\ &= \frac{\partial}{\partial y_t}\Big[e^{-\delta t}\ln(e^{rt}y_t)\Big] = \lambda \ \forall\forall t, \quad \gamma = 1 \end{aligned} \tag{27}$$

(where "$\forall\forall$" means "for almost all", i.e. for all but a set of Lebesgue measure zero). If (27) fails, then there are two positive-measure sets of times $\mathcal{T}_1$, $\mathcal{T}_2$ such that $v'(y_t) > v'(y_s)$ for $t \in \mathcal{T}_2$, $s \in \mathcal{T}_1$ (in turn implying $y_s > 0$ for $s \in \mathcal{T}_1$, since $v'(0) = \infty$). y is thus not optimal: a small reallocation from $\mathcal{T}_1$ to $\mathcal{T}_2$ increases U. If y does satisfy (27) almost everywhere, it is optimal, because $U(\cdot)$ is continuous and strictly quasiconcave: $U(a\tilde{x}+(1-a)\tilde{\tilde{x}}) > aU(\tilde{x})+(1-a)U(\tilde{\tilde{x}})$ for any pair of schedules $\tilde{x}$, $\tilde{\tilde{x}}$ that differ on a positive-measure set of times and for which $U(\tilde{x}) = U(\tilde{\tilde{x}})$. If an alternative feasible allocation $\tilde{y}$ would be superior to y, marginal reallocations from y toward $\tilde{y}$ would also be superior, which (27) rules out.

Taking the derivative and rearranging, we have

$$y_t = \lambda^{-\frac{1}{\gamma}} e^{\frac{r-r\gamma-\delta}{\gamma}t} \quad \forall\forall t. \tag{28}$$

Subjecting (28) to the budget constraint (3), we have

$$\int_0^\infty \lambda^{\frac{-1}{\gamma}} e^{\frac{r-r\gamma-\delta}{\gamma}t} dt = B. \tag{29}$$

If $\delta > r(1-\gamma)$, we find

$$\lambda^{-\frac{1}{\gamma}} = B\,\frac{r\gamma - r + \delta}{\gamma}\,\big(= B\alpha\big). \tag{30}$$

Substituting (30) into (28), and recalling that $x_t = e^{rt} y_t$, we have

$$x_t = B\alpha e^{(r-\alpha)t} \quad \forall\forall t. \tag{31}$$

Since the only right-continuous schedule x that satisfies (31) is that which equals the expression above for all t, this is the optimal schedule.

If $\delta \leq r(1-\gamma)$, no λ satisfies (29), so no budget-exhausting y satisfies (27). There is no optimal schedule.

Payoff to following the optimal private schedule

Given $\delta > r(1-\gamma)$, i.e. $\alpha > 0$, the payoff to following schedule (31) equals

$$U = \int_0^\infty e^{-\delta t} u\Big(B\alpha e^{(r-\alpha)t}\Big) dt,$$

which simplifies to

$$U = \begin{cases} \frac{B^{1-\gamma}}{1-\gamma}\alpha^{-\gamma}, & \gamma \neq 1; \\ \\ \frac{\delta \ln(B\delta) + r - \delta}{\delta^2}, & \gamma = 1. \end{cases} \tag{32}$$

A.2 Proof of Proposition 2

The payoff obtained given a loss of fraction w of the budget, without a spending rate requirement, equals (32) with $(1-w)B$ in place of B. Denote the resulting expression $U(B, w)$.

To find the payoff obtained under the spending rate requirement, substitute (5) with $\tilde{\alpha}$ in place of α into (2) and integrate to get

$$U_\delta(B, \tilde{\alpha}) \equiv \begin{cases} \frac{B^{1-\gamma}}{1-\gamma}\,\frac{\tilde{\alpha}^{1-\gamma}}{\tilde{\alpha}+\delta-\tilde{\delta}}, & \gamma \neq 1; \\ \\ \frac{\delta \ln(B\tilde{\delta}) + r - \tilde{\delta}}{\delta^2}, & \gamma = 1, \end{cases} \tag{33}$$

where, given $\tilde{\alpha}$,

$$\tilde{\delta} \equiv \tilde{\alpha}\gamma - r\gamma + r \tag{34}$$

is the time preference rate for which $\tilde{\alpha}$ is the optimal spending rate. Note that the integral is defined iff (7) holds.

Set $U_\delta(B, \tilde{\alpha}) = U(B, w)$ and solve for w.

A.3 Proof of Proposition 3

Consider a schedule x such that there are positive-measure sets of times $\mathcal{T}_1$, $\mathcal{T}_2$ with $\sup(\mathcal{T}_1) < \inf(\mathcal{T}_2)$, $x_t^L > 0\ \forall t \in \mathcal{T}_1$, and $x_t^H > 0\ \forall t \in \mathcal{T}_2$. We begin by showing that x is not an equilibrium.

If $e^{(r-\delta^L)t}(x_t^H + x_t^L)^{-\gamma}$ is not constant almost everywhere in $\mathcal{T}_1$, L prefers to reallocate his spending within $\mathcal{T}_1$. Likewise, if $e^{(r-\delta^H)t}(x_t^H + x_t^L)^{-\gamma}$ is not constant almost everywhere in $\mathcal{T}_2$, H prefers to reallocate her spending within $\mathcal{T}_2$. In either case, x is not an equilibrium.

Suppose therefore that

$$e^{(r-\delta^L)t}(x_t^H + x_t^L)^{-\gamma} = \lambda^L \ \ \forall\forall t \in \mathcal{T}_1, \tag{35}$$

$$e^{(r-\delta^H)t}(x_t^H + x_t^L)^{-\gamma} = \lambda^H \ \ \forall\forall t \in \mathcal{T}_2. \tag{36}$$

If x is an equilibrium, we must have

$$e^{(r-\delta^L)t}(x_t^H + x_t^L)^{-\gamma} \leq \lambda^L \ \ \forall\forall t \in \mathcal{T}_2. \tag{37}$$

Otherwise L prefers to marginally reallocate his spending from $\mathcal{T}_1$ to $\mathcal{T}_2$. Likewise, we must have

$$e^{(r-\delta^H)t}(x_t^H + x_t^L)^{-\gamma} \leq \lambda^H \ \ \forall\forall t \in \mathcal{T}_1, \tag{38}$$

or else H prefers to marginally reallocate her spending from $\mathcal{T}_2$ to $\mathcal{T}_1$. Multiplying both sides of (37) by $e^{(\delta^L-\delta^H)t}$ and both sides of (38) by $e^{(\delta^H-\delta^L)t}$, substituting for the left-hand sides by (36) and (35), and rearranging, we have

$$e^{(\delta^H-\delta^L)t} \leq \lambda^L/\lambda^H \ \ \forall\forall t \in \mathcal{T}_2,$$

$$e^{(\delta^H-\delta^L)t} \geq \lambda^L/\lambda^H \ \ \forall\forall t \in \mathcal{T}_1.$$

Since $\delta_H > \delta_L$ and $\mathcal{T}_2$ follows $\mathcal{T}_1$, these inequalities cannot simultaneously obtain. This completes the proof that x is not an equilibrium.

It follows that, in any equilibrium, there is a time t^o such that H is the sole funder $\forall\forall t < t^o$ and L is the sole funder $\forall\forall t > t^o$.

As in Proposition 1, each player i does best to spend such that the value of marginal allocations is equal across almost all times at which she spends. This implies that if x is an equilibrium,

$$x_t^H = \Lambda^H e^{(r-\alpha^H)t} \quad \forall\forall t < t^o, \qquad (39)$$
$$= 0 \quad \forall\forall t > t^o;$$
$$x_t^L = 0 \quad \forall\forall t < t^o, \qquad (40)$$
$$= \Lambda^L e^{(r-\alpha^L)t}, \quad \forall\forall t > t^o,$$

for some constants Λ^H, Λ^L.

From the budget constraints

$$\int_0^{t^o} \Lambda^H e^{-\alpha^H t} dt = B^H;$$
$$\int_{t^o}^{\infty} \Lambda^L e^{-\alpha^L t} dt = B^L,$$

we then have

$$\Lambda^H = B^H \alpha^H \left(1 - e^{-\alpha^H t^o}\right)^{-1}; \qquad (41)$$
$$\Lambda^L = B^L \alpha^L e^{-\alpha^L t^o}. \qquad (42)$$

An equilibrium collective schedule cannot exhibit a jump discontinuity at its t^o. If collective spending rose discontinuously at t^o, L would prefer a reallocation from just after t^o to just before. Likewise, if it fell discontinuously, H would prefer a reallocation from just before t^o to just after. We therefore have

$$\Lambda^H e^{(r-\alpha^H)t^o} = \Lambda^L e^{(r-\alpha^L)t^o}$$
$$\implies t^o = \frac{1}{\alpha^H - \alpha^L} \ln\left(\frac{\Lambda^H}{\Lambda^L}\right). \qquad (43)$$

Substituting (43) into (41)–(42) and simplifying, we get

$$\Lambda^H = B^H \alpha^H + B^L \alpha^L; \qquad (44)$$
$$\Lambda^L = B^L \alpha^L \left(1 + \frac{B^H \alpha^H}{B^L \alpha^L}\right)^{\frac{\alpha^L}{\alpha^H}}. \qquad (45)$$

Substituting (44)–(45) into (39)–(40) and (43) produces final expressions for x^H, x^L, and t^o.

We have shown that, if an equilibrium exists, it must take the form above. To verify that the profile (x^H, x^L) above is in fact an equilibrium, observe that, by construction, from (x^H, x^L), neither player prefers marginal reallocations across times during which he spends, or from times during which he spends to times during which he does not. Because $U^i(\cdot)$ is strictly quasiconcave for both i, neither player prefers a non-marginal alternative feasible $\tilde{x}^i$, fixing $\tilde{x}^{-i}$, either. x^L and x^H are mutual best responses.

A.4 Proof of Proposition 4

Preliminaries

Let $x^L(x^H)$ and $y^L(x^H)$ denote L's best-response individual schedule and allocation, respectively, to H's individual schedule x^H. The strict quasiconcavity of $U(\cdot)$ guarantees that if a best response exists, it is unique (up to measure-zero deviations). We will construct the unique form that an equilibrium $(x^H, x^L(x^H))$ must take if one exists and then show that the strategies are indeed mutual best responses.

Given x^H, let

$$\mathcal{T}^L(x^H) \equiv \{t : x_t^L(x^H) > 0\}, \quad \mathcal{T}^H(x^H) \equiv \{t : x_t^H > 0\}.$$

Note that the budget constraints

$$\int_{\mathcal{T}^L(x^H)} y_t^L(x^H)dt \leq B^L, \quad \int_{\mathcal{T}^H(x^H)} y_t^H dt \leq B^H$$

imply that $\mathcal{T}^L(x^H)$ and $\mathcal{T}^H(x^H)$, and thus their intersection, are measurable. Let

$$\mathcal{B}(x^H) \equiv \int_{\mathcal{T}^L(x^H)} y_t^L(x^H) \;\; (\leq B^H)$$

denote the budget that H allocates to $\mathcal{T}^L(x^H)$.

$y^L(x^H)$ must set the δ^L-discounted marginal utility of flow allocations equal $\forall\forall t \in \mathcal{T}^L(x^H)$. As in Proposition 1, this holds iff

$$x_t^H + x_t^L(x^H) = \Lambda^L(x^H)e^{(r-\alpha^L)t} \;\; \forall\forall t \in \mathcal{T}^L(x^H), \tag{46}$$

for the constant $\Lambda^L(x^H)$ satisfies the budget constraint

$$\int_{\mathcal{T}^L(x^H)} e^{-rt}(x_t^H + x_t^L(x^H))dt = \mathcal{B}(x^H) + B^L. \tag{47}$$

The resulting $\{x_t^H + x_t^L(x^H)\}_{t \in \mathcal{T}^L(x^H)}$ will implement the unique δ^L-optimal allocation of $\mathcal{B}(x^H) + B^L$ across $\mathcal{T}^L(x^H)$.

Let $\mathcal{T} \subset \mathcal{T}^L(x^H)$ denote a set of times such that

$$\int_{\mathcal{T}} e^{-rt}(x_t^H + X_t^L(x^H))dt = \mathcal{B}(x^H).$$

Such $\mathcal{T}$ must exist, by the continuity of total resource allocation with respect to time. If $\mathcal{B}(x^H) > 0$ any such $\mathcal{T}$ must have positive measure.

Suppose $\mathcal{B}(x^H) > 0$, and consider an $\tilde{x}^H$ such that

$$\tilde{x}_t^H = \begin{cases} x_t^H, & t \notin \mathcal{T}^L(x^H); \\ x_t^H + x_t^L(x^H), & t \in \mathcal{T}; \\ 0, & t \in \mathcal{T}^L(x^H) \backslash \mathcal{T}. \end{cases}$$

L will then still be able to achieve a collective schedule of $x_t^H + x_t^L(x^H)$, by adopting individual schedule $\tilde{x}^L$ with

$$\tilde{x}_t^L = \begin{cases} 0, & t \notin \mathcal{T}^L(x^H) \backslash \mathcal{T}; \\ x_t^H + x_t^L(x^H), & t \in \mathcal{T}^L(x^H) \backslash \mathcal{T}. \end{cases}$$

$(\tilde{x}^H, \tilde{x}^L)$ still implements the unique δ^L-optimal allocation of resources $\mathcal{B}(x^H) + B^L$ across $\mathcal{T}^L(x^H)$. Furthermore, because $\tilde{x}_t^H = x_t^H \; \forall t \notin \mathcal{T}^L(x^H)$, L still weakly prefers marginal spending within $\mathcal{T}^L(x^H)$ to marginal spending outside $\mathcal{T}^L(x^H)$. Given $\tilde{x}^H$, therefore, L does indeed best respond with $\tilde{x}^L$; $\tilde{x}^L = x^L(\tilde{x}^H)$.

$\tilde{x}^H$ thus induces the same collective schedule as x^H. However, $\mathcal{B}(\tilde{x}^H) = 0$. We have found that, for any feasible x^H, there is a feasible $\tilde{x}^H$ such that

$$\mathcal{B}(\tilde{x}^H) = 0 \tag{48}$$

and H is indifferent between x^H and $\tilde{x}^H$ given L's best response.

Given a feasible x^H satisfying (48), let

$$\hat{\mathcal{T}}(x^H) \equiv \{t : x_t^H > \Lambda^L(x^H)e^{(r-\alpha^H)t}\},$$

$$\hat{\mathcal{B}}(x^H) \equiv \int_{\hat{\mathcal{T}}(x^H)} y_t^H dt.$$

Observe by the reasoning of Proposition 1 that there is a (near-)unique and δ^H-optimal feasible $\tilde{x}^H$, under the condition $\mathcal{T}^H(\tilde{x}^H) = \mathcal{T}^H(x^H)$, such that

$$\tilde{x}_t^H = \begin{cases} 0, & t \notin \mathcal{T}^H(x^H); \\ \max(\Lambda^H(\tilde{x}^H)e^{(r-\alpha^H)t}, \Lambda^L(\tilde{x}^H)e^{(r-\alpha^L)t}), & t \in \mathcal{T}^H(x^H), \end{cases} \tag{49}$$

where $\Lambda^L(\tilde{x}^H) = \Lambda^L(x^H)$ is given by (46)–(47) and

$$\Lambda^H(\tilde{x}^H) \equiv \begin{cases} \Lambda^H : \int_{\hat{\mathcal{T}}(x^H)} \Lambda^H e^{-\alpha^H t} dt = \hat{\mathcal{B}}(x^H) \\ \qquad \Longrightarrow \hat{\mathcal{B}}(x^H) \big/ \int_{\hat{\mathcal{T}}(x^H)} e^{-\alpha^H t} dt, & \hat{\mathcal{B}}(x^H) > 0; \\ \Lambda^L(\tilde{x}^H), & \hat{\mathcal{B}}(x^H) = 0. \end{cases}$$

That is, H always does best to allocate δ^H-optimally across the periods during which she spends, at least subject to the constraint that such a shift does not render her spending at a time low enough that L responds by altering his own schedule.

Consider a feasible allocation y^H (and corresponding x^H) satisfying (48) and (49). We must have

$$y_t^H = \begin{cases} \Lambda^H(x^H) e^{-\alpha^H t}, & t \in \mathcal{T}^H(x^H) \cap [0, \overline{q}]; \\ \Lambda^L(x^H) e^{-\alpha^L t}, & t \in \mathcal{T}^H(x^H) \cap [\overline{q}, \infty); \end{cases} \tag{50}$$

$$y_t^L(x^H) = \quad \Lambda^L(x^H) e^{-\alpha^L t}, \quad t \in \mathcal{T}^L(x^H),$$

where

$$\overline{q}(x^H) \equiv \ln\Big(\frac{\Lambda^L(x^H)}{\Lambda^H(x^H)}\Big) / (\alpha^L - \alpha^H)$$

denotes the time q at which $\Lambda^H(x^H) e^{-\alpha^H q} = \Lambda^L(x^H) e^{-\alpha^L q}$.

Define

$$\underline{Q}(q; x^H) \equiv \int_{\mathcal{T}^L(x^H) \cap [0,q)} \Lambda^L(x^H) e^{-\alpha^L t}, \tag{51}$$

$$\overline{Q}(q; x^H) \equiv \int_{\mathcal{T}^H(x^H) \cap [\max(\overline{q}, q), \infty)} \Lambda^L(x^H) e^{-\alpha^L t}.$$

Since $\underline{Q}(q; x^H)$ weakly and continuously increases in q from zero to B^L, and $\overline{Q}(q; x^H)$ weakly and continuously decreases in q from a nonnegative value to zero, there exists a (not necessarily unique) q^* such that

$$\underline{Q}(q^*; x^H) = \overline{Q}(q^*; x^H).$$

In equilibrium, q^* is a time such that the amount (in present-value terms) that L allocates δ^L-optimally before q^* equals the amount that H allocates δ^L-optimally after q^*.

Choosing some such q^*, now consider the allocation

$$\tilde{y}_t^H = \begin{cases} \Lambda^L(x^H) e^{-\alpha^L t}, & t \in \mathcal{T}^L(x^H) \cap [0, q^*); \\ 0, & t \in \mathcal{T}^H(x^H) \cap [\max(\overline{q}(x^H), q^*), \infty); \\ y_t^H, & \text{elsewhere.} \end{cases}$$

Observe that $\Lambda^L(\tilde{x}^H) = \Lambda^L(x^H)$ and $\Lambda^H(\tilde{x}^H) = \Lambda^H(x^H)$, and thus that $\overline{q}(\tilde{x}^H) = \overline{q}(x^H)$.

It follows from (51) that L's (near-)unique best response to a shift from y^H to $\tilde{y}^H$ is to shift his spending from $\mathcal{T}^L(x^H) \cap [0, q^*)$ to $\mathcal{T}^H(x^H) \cap [\max(\overline{q}(x^H), q^*), \infty)$, leaving his spending elsewhere unchanged; this alone maintains (46). It likewise follows from (51) that $\tilde{y}^H$ is affordable for H, and that it induces the same collective allocation as y^H. Finally, $\sup(\mathcal{T}^H(\tilde{x}^H)) \leq \max(\overline{q}(\tilde{x}^H), q^*)$, so

$$T(\tilde{x}^H) \equiv [\inf(\mathcal{T}^L(\tilde{x}^H)), \sup(\mathcal{T}^H(\tilde{x}^H))] \text{ is bounded and} \qquad (52)$$
$$\tilde{x}^H_t = \Lambda^H(\tilde{x}^H) e^{(r-\alpha^H)t} \;\; \forall t \in T(\tilde{x}^H),$$

with the second observation following from (49)–(50) and the fact that $\overline{q}(\tilde{x}^H) = \overline{q}(x^H)$. Note that $\inf(\mathcal{T}^L(x^L(\tilde{x}^H))) \geq q^*$, so $T(\tilde{x}^H)$ is of positive measure only if $\max(T(\tilde{x}^H)) = \overline{q}(\tilde{x}^H)$.

We have now shown that, for any feasible x^H satisfying (48) and (49), there is a feasible $\tilde{x}^H$ satisfying (48) and (52) such that H is indifferent between x^H and $\tilde{x}^H$ given L's best response.

Without loss of generality, let us define $x^L(\cdot)$ such that

$$\int_{T(x^H)} e^{-rt} x^L_t(x^H) dt > 0 \;\; \forall x^H : \inf(\mathcal{T}^L(x^H)) < \sup(\mathcal{T}^H(x^H)). \qquad (53)$$

That is, let us not say that L ever responds to x^H by spending at a measure-zero set of times before he begins to allocate positive resources.

If

$$\int_{T(x^H)} e^{-rt} x^H_t dt = 0, \qquad (54)$$

x^H is a measure-zero deviation from an alternative strategy for H, $\tilde{x}^H$ with

$$\tilde{x}^H_t = \begin{cases} x^H_t, & t < \inf(\mathcal{T}^L(x^H)); \\ 0, & t \geq \inf(\mathcal{T}^L(x^H)), \end{cases} \qquad (55)$$

such that the schedule $(\tilde{x}^H, x^L(\tilde{x}^H))$ is polarized in the sense that

$$\exists \tilde{t}(\tilde{x}^H) \;:\; x^L_t(\tilde{x}^H) = 0 \;\; \forall t < \tilde{t}(\tilde{x}^H), \quad \tilde{x}^H_t = 0 \;\; \forall t \geq \tilde{t}(\tilde{x}^H). \qquad (56)$$

Of course, $x^L(\tilde{x}^H) = x^L(x^H)$ up to measure-zero deviations and, given these best responses, H is indifferent between x^H and $\tilde{x}^H$.

The next step of the proof is to show that for any feasible x^H satisfying (48) and (52) but not (54), there is a feasible $\tilde{x}^H$, which H in equilibrium strictly prefers to

x^H, satisfying all three conditions. It will then follow that, in equilibrium, x^H must be at most a zero-measure deviation from an $\tilde{x}^H$ satisfying (56). We will then be able to determine the (near-)uniquely optimal x^H in this class, and thus the equilibrium as a whole.

From here we will proceed differently depending on the value of γ.

$\gamma > 1$ case

Consider a feasible allocation y^H (and corresponding x^H) satisfying (48) and (52) but not (54). Recall that this implies that $\max(T(x^H)) \leq \overline{q}(x^H)$.

Define

$$\underline{Z}(z; x^H) \equiv \int_{\mathcal{T}^L(x^H)\cap[0,z)} e^{-\alpha^L t} dt, \tag{57}$$

$$\overline{Z}(z; x^H) \equiv \int_{\mathcal{T}^H(x^H)\cap[z,\overline{q}(x^H))} e^{-\alpha^L t} dt. \tag{58}$$

Since $\overline{Z}(z; x^H) - \underline{Z}(z; x^H)$ strictly and continuously decreases in z from a positive value at $z = 0$ to a negative (by (53)) value at $z = \overline{q}(x^H)$, there is a unique $z^* \in (0, \overline{q}(x^H))$ such that $\underline{Z}(z^*; x^H) = \overline{Z}(z^*; x^H)$. Let

$$\begin{aligned} \underline{T}(x^H) &\equiv \mathcal{T}^L(x^H) \cap [0, z^*), \\ \overline{T}(x^H) &\equiv \mathcal{T}^H(x^H) \cap [z^*, \overline{q}(x^H)). \end{aligned} \tag{59}$$

It follows from the fact that $\sup(\mathcal{T}^H(x^H)) \leq \overline{q}(x^H)$ that $y^H_t \geq \Lambda^L(x^H) e^{-\alpha^L t}$ $\forall t \in \mathcal{T}^H(x^H)$.

Choose $\epsilon > 0$, and partition $\overline{T}(x^H)$ into (not necessarily nonempty) elements

$$\overline{T}_{j,\epsilon}(x^H) \equiv \overline{T}(x^H) \cap [z^* + j - \epsilon, z^* + j) \tag{60}$$

for all $j \in \epsilon\mathbb{N}$, where $\epsilon\mathbb{N}$ denotes $\{\epsilon, 2\epsilon, ...\}$. Also, define

$$\underline{t}_j(x^H) \equiv \min\left\{q : \int_{\underline{T}(x^H)\cap[0,q)} e^{-\alpha^L t} dt = \int_{\overline{T}(x^H)\cap[z^*,z^*+j)} e^{-\alpha^L t} dt\right\}, \tag{61}$$

$$\underline{T}_{j,\epsilon}(x^H) \equiv \underline{T}(x^H) \cap [\underline{t}_{j-\epsilon}(x^H), \underline{t}_j(x^H)), \tag{62}$$

$$j(t, \epsilon; x^H) \equiv j \ : \ t \in \underline{T}_{j,\epsilon}(x^H) \cup \overline{T}_{j,\epsilon}(x^H). \tag{63}$$

That is, $j(t, \epsilon; x^H)$ is the element of $\epsilon\mathbb{N}$ such that t lies in the element of the partitioned $\overline{T}(x^H)$ whose supremum is j, or the corresponding element of the partitioned $\underline{T}(x^H)$.

Let

$$S(x^H) \equiv \left\{s \in \overline{T}(x^H) \ : \ \lim_{\epsilon\to 0} \left(\underline{t}_{j(s,\epsilon;x^H)}(x^H) - \underline{t}_{j(s,\epsilon;x^H)-\epsilon}(x^H)\right) > 0\right\}. \tag{64}$$

It follows from (61) that, for all $s \in S(x^H)$,

$$\exists \phi > 0 : \int_{\underline{T}(x^H)\cap[0,\underline{t}_s(x^H))} e^{-\alpha^L t}dt = \int_{\underline{T}(x^H)\cap[0,\underline{t}_s(x^H)+\phi)} e^{-\alpha^L t}dt = \int_{\overline{T}(x^H)\cap[z^*,s)} e^{-\alpha^L t}dt. \tag{65}$$

Given $s \in S(x^H)$, let ϕ_s denote the supremum ϕ satisfying (65). Thus, for each $s \in S(x^H)$, there is a maximal subinterval $\Phi(s) \equiv [\underline{t}_s(x^H), \phi_s)$ of $[0, z^*)$ such that $\mu(\underline{T}(x^H) \cap \Phi(s)) = 0$. Because any interval can be partitioned into at most countably many subintervals, there are at most countably many such Φ. Furthermore, for each Φ, we must have

$$\mu\big(\{s : \Phi(s) = \Phi\}\big) = 0;$$

otherwise the integral on the right-hand side of (65) could not be equal for all such s. Therefore $\mu(S(x^H)) = 0$.

It follows that, given any feasible x^H satisfying (48) and (52), there is a corresponding feasible $\tilde{x}^H$ with

$$\tilde{x}^H_t = \begin{cases} x^H_t, & t \notin S(x^H); \\ 0, & t \in S(x^H), \end{cases}$$

also satisfying (48) and (52), but for which we also have

$$S(\tilde{x}^H) = \emptyset. \tag{66}$$

Of course $x^L(\tilde{x}^H) = x^L(x^H)$ almost everywhere, and H is indifferent between x^H and $\tilde{x}^H$ given L's best responses.

Consider an allocation y^H (and corresponding x^H) satisfying (48), (52), and (66) but not (54). Then, given a choice of interval-length ϵ, consider the allocation $\tilde{y}^H(\epsilon)$ (and corresponding $\tilde{x}^H(\epsilon)$) with

$$\tilde{y}^H_t(\epsilon) = \begin{cases} y^H_t, & t \notin \overline{T}(x^H) \cup \underline{T}(x^H); \\ \Lambda^H(x^H) e^{(\alpha^L-\alpha^H)(z^*+j(t,\epsilon;x^H))-\alpha^L(\underline{t}_{j(t,\epsilon;x^H)}(x^H)+\epsilon)}, & t \in \underline{T}(x^H); \\ 0, & t \in \overline{T}(x^H). \end{cases}$$

Note that $\tilde{x}^H(\epsilon)$ satisfies (54) for all ϵ.

To demonstrate that $\tilde{y}^H(\epsilon)$ is feasible, let us show that its allocation to each $\underline{T}_{j,\epsilon}(x^H)$ is weakly (and indeed strictly) less than y^H's allocation to the corresponding $\overline{T}_{j,\epsilon}(x^H)$. From (61) and (62), we have

$$\int_{\underline{T}_{j,\epsilon}(x^H)} e^{-\alpha^L t}dt = \int_{\overline{T}_{j,\epsilon}(x^H)} e^{-\alpha^L t}dt. \tag{67}$$

Observe that $t < \underline{t}_j(x^H)\ \forall t \in \underline{T}_{j,\epsilon}(x^H)$ and $t \geq z^* + j - \epsilon\ \forall t \in \overline{T}_{j,\epsilon}(x^H)$. Also,

$$y_t^H \geq \Lambda^H(x^H)e^{-\alpha^H(z^*+j)} \quad \forall t \in \overline{T}_{j,\epsilon}(x^H).$$

Thus (67) gives

$$\int_{\underline{T}_{j,\epsilon}(x^H)} e^{-\alpha^L \underline{t}_j(x^H)} dt \leq \int_{\overline{T}_{j,\epsilon}(x^H)} e^{-\alpha^L(z^*+j-\epsilon)} dt \tag{68}$$

$$\implies \int_{\underline{T}_{j,\epsilon}(x^H)} \Lambda^H(x^H) e^{(\alpha^L-\alpha^H)(z^*+j)-\alpha^L(\underline{t}_j(x^H)+\epsilon)} dt \leq \int_{\overline{T}_{j,\epsilon}(x^H)} \Lambda^H(x^H)e^{-\alpha^H(z^*+j)}dt$$
$$< \int_{\overline{T}_{j,\epsilon}(x^H)} \Lambda^H(x^H)e^{-\alpha^H t}dt. \tag{69}$$

Summing across $j \in \epsilon\mathbb{N}$, it follows that, since y^H is feasible, $\tilde{y}^H(\epsilon)$ is also feasible.

Now let us show that, for sufficiently small ϵ, H strictly prefers $\tilde{y}^H(\epsilon)$ to y^H given L's best responses.

First, for any ϵ, we can decompose the move from y^H to $\tilde{y}^H(\epsilon)$ into a sequence of shifts from $\{y_t^H\}$ to $\{\tilde{y}_t^H(\epsilon)\}$ for $t \in \underline{T}_{j,\epsilon}(x^H) \cup \overline{T}_{j,\epsilon}(x^H)$ for each j, with the original allocation maintained elsewhere. That is, H can shift spending back from $\overline{T}(x^H)$ to $\underline{T}(x^H)$ by shifting spending back from $\overline{T}_{j,\epsilon}(x^H)$ to $\underline{T}_{j,\epsilon}(x^H)$ for each j. Each shift will be affordable, as shown by (69). Furthermore, from (61) and (62) we see that, in equilibrium, L will (up to measure-zero deviations) respond to each shift by shifting his own allocated funds from $\underline{T}_{j,\epsilon}(x^H)$ to $\overline{T}_{j,\epsilon}(x^H)$, leaving his spending elsewhere unchanged; this alone maintains condition (46) for some Λ^L. Finally, observe that the shift must increase flow spending and thus utility throughout $\underline{T}_{j,\epsilon}(x^H)$ and decrease it throughout $\overline{T}_{j,\epsilon}(x^H)$.

H's net utility gain from shift j is thus bounded below by

$$\frac{1}{1-\gamma}\Bigg[\int_{\underline{T}_{j,\epsilon}(x^H)} e^{-\delta^H \underline{t}_j(x^H)}\Big(\big(\Lambda^H(x^H)e^{(\alpha^L-\alpha^H)(z^*+j)-\alpha^L(\underline{t}_j(x^H)+\epsilon)+r\underline{t}_j(x^H)}\big)^{1-\gamma}$$
$$-\big(\Lambda^L(x^H)e^{(r-\alpha^L)\underline{t}_{j-\epsilon}(x^H)}\big)^{1-\gamma}\Big)dt \tag{70}$$
$$+\int_{\overline{T}_{j,\epsilon}(x^H)} e^{-\delta^H(z^*+j-\epsilon)}\Big(\big(\Lambda^L(x^H)e^{(r-\alpha^L)(z^*+j)}\big)^{1-\gamma} - \big(\Lambda^H(x^H)e^{(r-\alpha^H)(z^*+j-\epsilon)}\big)^{1-\gamma}\Big)dt\Bigg].$$

From (67), the fact that $t \geq \underline{t}_{j-\epsilon}(x^H)\ \forall t \in \underline{T}_{j,\epsilon}(x^H)$, and the fact that $t < z^* + j\ \forall t \in \overline{T}_{j,\epsilon}(x^H)$, we have

$$\int_{\overline{T}_{j,\epsilon}(x^H)} e^{-\alpha^L(z^*+j)}dt \leq \int_{\underline{T}_{j,\epsilon}(x^H)} e^{-\alpha^L \underline{t}_{j-\epsilon}(x^H)}dt. \tag{71}$$

After rearranging (70) and making the substitution from (71), we see that H's net utility gain from shift j is further bounded below by

$$\frac{1}{\gamma-1}\big(\Lambda^L(x^H)e^{(r-\alpha^L)\underline{t}_{j-\epsilon}(x^H)}\big)^{1-\gamma}\int_{\overline{T}_{j,\epsilon}(x^H)} e^{-\alpha^L(z^*+j)}dt \tag{72}$$

$$\Bigg[\bigg(1-\Big(\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)}e^{(\alpha^L-\alpha^H)(z^*+j)+(r-\alpha^L)(\underline{t}_j(x^H)-\underline{t}_{j-\epsilon}(x^H))-\alpha^L\epsilon}\Big)^{1-\gamma}\bigg)e^{\alpha^L\underline{t}_{j-\epsilon}(x^H)-\delta^H\underline{t}_j(x^H)}$$

$$-\bigg(1-\Big(\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)}e^{(\alpha^L-\alpha^H)(z^*+j)-(r-\alpha^H)\epsilon}\Big)^{1-\gamma}\bigg)$$

$$e^{(1-\gamma)(r-\alpha^L)(z^*+j-\underline{t}_{j-\epsilon}(x^H))-\delta^H(z^*+j-\epsilon)+\alpha^L(z^*+j)}\Bigg].$$

By (64) and (66),

$$\lim_{\epsilon\to 0}\underline{t}_{j(t,\epsilon;x^H)}(x^H)=\lim_{\epsilon\to 0}\underline{t}_{j(t,\epsilon;x^H)-\epsilon}(x^H)=\underline{t}_{t-z^*}(x^H)\ \ \forall t\in\overline{T}(x^H). \tag{73}$$

Also, by (60) and (63),

$$\lim_{\epsilon\to 0} j(t,\epsilon;x^H)=t-z^*\ \ \forall t\in\overline{T}(x^H). \tag{74}$$

Furthermore, $\underline{t}_{j(t,\epsilon;x^H)}(x^H)-\underline{t}_{j(t,\epsilon;x^H)-\epsilon}(x^H)$ is uniformly bounded by z^*, and $j(t,\epsilon;x^H)-(t-z^*)$ by $\overline{q}(x^H)-z^*$, for $t\in\overline{T}(x^H)$; so these two convergences are uniform throughout $\overline{T}(x^H)$. It follows that

$$\lim_{\epsilon\to 0}\Big[1-\Big(\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)}e^{(\alpha^L-\alpha^H)(z^*+j(t,\epsilon;x^H))+(r-\alpha^L)(\underline{t}_{j(t,\epsilon;x^H)}(x^H)-\underline{t}_{j(t,\epsilon;x^H)-\epsilon}(x^H))-\alpha^L\epsilon}\Big)^{1-\gamma}\Big] \tag{75}$$

$$=\lim_{\epsilon\to 0}\Big[1-\Big(\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)}e^{(\alpha^L-\alpha^H)(z^*+j(t,\epsilon;x^H))-(r-\alpha^H)\epsilon}\Big)^{1-\gamma}\Big] \tag{76}$$

$$=1-\Big(\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)}e^{(\alpha^L-\alpha^H)t}\Big)^{1-\gamma} \tag{77}$$

for all $t\in\overline{T}(x^H)$, and that the convergence of (75) and (76) to (77) is uniform throughout $\overline{T}(x^H)$.

By (48) and (52),

$$\Lambda^H(x^H)e^{-\alpha^H t}>\Lambda^L(x^H)e^{-\alpha^L t}\ \ \forall t\in[z^*,\overline{q}(x^H)), \tag{78}$$

so $\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)}e^{(\alpha^L-\alpha^H)t}>1$ for all such t. By our assumption of $\gamma>1$, term (77) is positive for all $t\in\overline{T}(x^H)$.

The net utility gain for H from the shift from y^H to $\tilde{y}^H(\epsilon)$—the sum of (72) across $j \in \epsilon\mathbb{N}$—equals

$$\frac{1}{\gamma-1}\int_{\overline{T}(x^H)} \left(\Lambda^L(x^H)e^{(r-\alpha^L)\underline{t}_{j(t,\epsilon;x^H)-\epsilon}(x^H)}\right)^{1-\gamma} e^{-\alpha^L(z^*+j(t,\epsilon;x^H))}$$

$$\Bigg[\Bigg(1-\Big(\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)}e^{(\alpha^L-\alpha^H)(z^*+j(t,\epsilon;x^H))+(r-\alpha^L)(\underline{t}_{j(t,\epsilon;x^H)}(x^H)-\underline{t}_{j(t,\epsilon;x^H)-\epsilon}(x^H))-\alpha^L\epsilon}\Big)^{1-\gamma}\Bigg)$$

$$e^{\alpha^L\underline{t}_{j(t,\epsilon;x^H)-\epsilon}(x^H)-\delta^H\underline{t}_{j(t,\epsilon;x^H)}(x^H)}$$

$$-\Bigg(1-\Big(\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)}e^{(\alpha^L-\alpha^H)(z^*+j(t,\epsilon;x^H))-(r-\alpha^H)\epsilon}\Big)^{1-\gamma}\Bigg)$$

$$e^{(1-\gamma)(r-\alpha^L)(z^*+j(t,\epsilon;x^H)-\underline{t}_{j(t,\epsilon;x^H)-\epsilon}(x^H))-\delta^H(z^*+j(t,\epsilon;x^H)-\epsilon)+\alpha^L(z^*+j(t,\epsilon;x^H))}\Bigg]dt.$$

From (73) to (78), this converges, as $\epsilon \to 0$, to a value strictly greater than

$$\frac{\Lambda^L(x^H)^{1-\gamma}}{\gamma-1}\int_{\overline{T}(x^H)}\left(1-\Big(\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)}e^{(\alpha^L-\alpha^H)t}\Big)^{1-\gamma}\right)\left(e^{(\delta^L-\delta^H)\underline{t}_{t-z^*}(x^H)}-e^{(\delta^L-\delta^H)t}\right)dt, \tag{79}$$

which is positive. Therefore the total net utility gain is positive; H strictly prefers $\tilde{y}^H(\epsilon)$, for a small ϵ, to y^H, given L's best response. An equilibrium x^H must therefore be at most a measure-zero deviation of one satisfying the polarization condition (56).

We will now find the x^H that, among those satisfying (56) for some $\tilde{t}(x^H)$, uniquely maximizes H's utility given L's best response.

Because L invests his resources until $\tilde{t}(x^H)$ and subsequently allocates them δ^L-optimally, we have

$$x_t^L(x^H) = \begin{cases} 0, & t < \tilde{t}(x^H); \\ \Lambda^L(x^H)e^{(r-\alpha^L)t}, & t \geq \tilde{t}(x^H), \end{cases} \tag{80}$$

where $\Lambda^L(x^H)$ satisfies

$$\int_{\tilde{t}(x^H)}^{\infty} \Lambda^L(x^H)e^{-\alpha^L t}dt = B^L \quad \Longrightarrow \quad \Lambda^L(x^H) = B^L\alpha^L e^{\alpha^L\tilde{t}(x^H)}. \tag{81}$$

Likewise, fixing $\tilde{t}$, denote H's most preferred x^H satisfying (56) with $\tilde{t}(x^H) = \tilde{t}$ by $x^H[\tilde{t}]$. $x^H[\tilde{t}]$ must spend H's resources δ^H-optimally up to $\tilde{t}$. Spending δ^H-optimally up to arbitrarily high $\tilde{t}$ will not be compatible with (52), and thus not with (56), in equilibrium; L will eventually prefer spending to waiting until $\tilde{t}$. Nevertheless, we will find the $\tilde{t}$ that would be optimal for H if H *could* spend δ^H-optimally up to $\tilde{t}$, instead of eventually having to switch to a δ^L-optimal schedule (as in (50)). We will then see that spending δ^H-optimally up to the optimal threshold time, t^*, is indeed compatible with (52) and thus (56) in equilibrium.

So we have

$$x_t^H[\tilde{t}] = \begin{cases} \Lambda^H[\tilde{t}]e^{(r-\alpha^H)t}, & t < \tilde{t}; \\ 0, & t \geq \tilde{t}, \end{cases} \tag{82}$$

where $\Lambda^H[\tilde{t}]$ $(= \Lambda^H(x^H[\tilde{t}]))$ satisfies

$$\int_0^{\tilde{t}} \Lambda^H[\tilde{t}]e^{-\alpha^H t}dt = B^H \quad \Longrightarrow \quad \Lambda^H[\tilde{t}] = \frac{B^H\alpha^H}{1-e^{-\alpha^H\tilde{t}}}. \tag{83}$$

$(x^H[\tilde{t}], x^L(x^H[\tilde{t}]))$ delivers H payoff

$$\frac{1}{1-\gamma}\Bigg[\int_0^{\tilde{t}} e^{-\delta^H t}\bigg(\Big(B^H\alpha^H\big(1-e^{-\alpha^H\tilde{t}}\big)^{-1}e^{(r-\alpha^H)t}\Big)^{1-\gamma}\bigg)dt$$

$$+\int_{\tilde{t}}^{\infty} e^{-\delta^H t}\bigg(\Big(B^L\alpha^L e^{\alpha^L\tilde{t}}e^{(r-\alpha^L)t}\Big)^{1-\gamma}\bigg)\Bigg]dt$$

$$= \frac{1}{1-\gamma}\Big[(B^H)^{1-\gamma}(\alpha^H)^{-\gamma}\big(1-e^{-\alpha^H\tilde{t}}\big)^{\gamma} + \frac{(B^L\alpha^L)^{1-\gamma}}{\delta^H+\alpha^L-\delta^L}e^{-\gamma\alpha^H\tilde{t}}\Big].$$

From the first order condition in $\tilde{t}$, we find a unique maximum at

$$\tilde{t} = t^* \equiv \ln\Big(\frac{B^H\alpha^H}{B^L\alpha^L}\eta + 1\Big)/\alpha^L. \tag{84}$$

Substituting (84) into (83) and (81), we find that the δ^H-optimal allocation rate approaching t^*, and the δ^L-optimal allocation rate at t^*, are respectively

$$\Lambda^H[t^*]e^{-\alpha^H t^*} = B^L\alpha^L/\eta, \tag{85}$$

$$\Lambda^L(x^H[t^*])e^{-\alpha^L t^*} = B^L\alpha^L.$$

Since $\eta < 1$, spending impatient-optimally up to t^* is compatible with (52) in equilibrium, as promised. $x^{H*} \equiv x^H[t^*]$ is thus the unique equilibrium strategy for H among those satisfying (56), and a near-unique equilibrium strategy for H overall.

The proof is completed under "last steps" below.

$\gamma = 1$ case

Follow the proof of the $\gamma > 1$ case up to (69). By the reasoning preceding (70), we can decompose the spending shift from $\overline{T}(x^H)$ to $\underline{T}(x^H)$, which constitutes the shift from y^H to $\tilde{y}^H(\epsilon)$, into shifts from $\overline{T}_{j,\epsilon}(x^H)$ to $\underline{T}_{j,\epsilon}(x^H)$ for each $j \in \epsilon\mathbb{N}$; and having done so, H's net utility loss from shift j is bounded above by

$$\int_{\underline{T}_{j,\epsilon}(x^H)} e^{-\delta^H \underline{t}_{j,\epsilon}(x^H)} \Big(\ln \big(\Lambda^L(x^H) e^{(r-\delta^L)\underline{t}_{j-\epsilon,\epsilon}(x^H)}\big) - \ln \big(\Lambda^H(x^H) e^{(\delta^L-\delta^H)(z^*+j)-\delta^L(\underline{t}_{j,\epsilon}(x^H)+\epsilon)+r\underline{t}_{j,\epsilon}(x^H)}\big)\Big) dt$$
$$+ \int_{\overline{T}_{j,\epsilon}(x^H)} e^{-\delta^L(z^*+j-\epsilon)} \Big(\ln \big(\Lambda^H(x^H) e^{(r-\delta^H)(z^*+j-\epsilon)}\big) - \ln \big(\Lambda^L(x^H) e^{(r-\delta^L)(z^*+j)}\big)\Big) dt. \tag{86}$$

After rearranging, and by (71), this implies that that H's total net utility loss across all j is bounded above by

$$\int_{\overline{T}(x^H)} e^{-\delta^L(z^*+j(t,\epsilon;x^H))} \Big[e^{\delta^L \underline{t}_{j(t,\epsilon;x^H)-\epsilon}(x^H) - \delta^H \underline{t}_{j(t,\epsilon;x^H)}(x^H)} \tag{87}$$
$$\Big(\ln\Big(\frac{\Lambda^L(x^H)}{\Lambda^H(x^H)}\Big) + r\big(\underline{t}_{j(t,\epsilon;x^H)-\epsilon}(x^H) - \underline{t}_{j(t,\epsilon;x^H)}(x^H)\big) + (\delta^H-\delta^L)(z^*+j(t,\epsilon;x^H)) + \delta^L\epsilon \Big)$$
$$- e^{(\delta^L-\delta^H)(z^*+j(t,\epsilon;x^H))-\delta^H\epsilon} \Big(\ln\Big(\frac{\Lambda^L(x^H)}{\Lambda^H(x^H)}\Big) + (\delta^H-\delta^L)(z^*+j(t,\epsilon;x^H)) + (r-\delta^H)\epsilon \Big) \Big] dt.$$

By the uniform convergences of (73) and (74), (87) converges to zero as $\epsilon \to 0$. Thus, for any $\ell > 0$, $\exists \bar{\epsilon} > 0$ such that

$$\Big| U^H\big((x^H, x^L(x^H))\big) - U^H\big((\tilde{x}^H(\epsilon), x^L(\tilde{x}^H(\epsilon)))\big) \Big| < \ell \;\; \forall \epsilon < \bar{\epsilon}. \tag{88}$$

Observe that, as $\epsilon \to 0$, $\tilde{x}^H(\epsilon)$ converges uniformly throughout $[0, \infty)$ to a spending schedule we might denote $\tilde{x}^H(0)$, which satisfies (48), (52), (66), and (54), and such that, by (88), H is indifferent between $\tilde{x}^H(0)$ and x^H given L's best response.

Consider the class of x^H satisfying (48), (52), (66), and (54). By the reasoning of (80)–(83), the (near-unique) optimal x^H for H in this class, given L's best response—which we may denote x^{H*}—delivers H payoff

$$\int_0^{\tilde{t}} e^{-\delta^H t} \ln\Big(\frac{B^H\delta^H}{1-e^{-\delta^H\tilde{t}}} e^{(r-\delta^H)t}\Big) dt + \int_{\tilde{t}}^{\infty} e^{-\delta^H t} \ln\Big(B^L\delta^L e^{\delta^L\tilde{t}} e^{(r-\delta^L)t}\Big) dt$$
$$= \frac{e^{-\delta^H\tilde{t}}}{\delta^H}\Big(\ln(B^L\delta^L) - \frac{\delta^L}{\delta^H} + \delta^H\tilde{t} + 1\Big) + \frac{e^{-\delta^H\tilde{t}}-1}{\delta^H} \ln\Big(\frac{1-e^{-\delta^H\tilde{t}}}{B^H\delta^H}\Big) - \frac{1}{\delta^H} + \frac{r}{(\delta^H)^2}$$

for some $\tilde{t}$. From the first order condition in $\tilde{t}$, we find a unique maximum at

$$\tilde{t} = t^* \equiv \ln\Big(1 + \frac{B^H\delta^H}{B^L\delta^L} e^{\frac{\delta^L}{\delta^H}-1}\Big)/\delta^H. \tag{89}$$

Substituting (89) into (83) and substituting both terms into (82), we obtain our expression for x^{H*}.

The proof is completed under "last steps" below.

$\gamma < 1$ case

Pick up from the end of the "Preliminaries" section above.

Consider a feasible allocation y^H (and corresponding x^H) satisfying (48) and (52) but not (54). Define

$$\hat{\underline{Z}}(z; x^H) \equiv \int_{\mathcal{T}^H(x^H) \cap [0,z)} e^{-\alpha^L t} dt,$$

$$\hat{\overline{Z}}(z; x^H) \equiv \int_{\mathcal{T}^L(x^H) \cap [z, \overline{q}(x^H))} e^{-\alpha^L t} dt.$$

Since $\hat{\overline{Z}}(z; x^H) - \hat{\underline{Z}}(z; x^H)$ strictly and continuously decreases in z from a positive value at $z = 0$ to a negative value at $z = \overline{q}(x^H)$, there is a unique $z^* \in (0, \overline{q}(x^H))$ such that $\underline{Z}(z^*; x^H) = \overline{Z}(z^*; x^H)$. Let

$$\hat{\underline{T}}(x^H) \equiv \mathcal{T}^H(x^H) \cap [0, z^*),$$

$$\hat{\overline{T}}(x^H) \equiv \mathcal{T}^L(x^H) \cap [z^*, \overline{q}(x^H)).$$

It follows from (48) that $y_t^H \geq \Lambda^L(x^H) e^{-\alpha^L t} \; \forall t \in \mathcal{T}^H(x^H)$.

Choose $\epsilon > 0$. Partition $\hat{\overline{T}}(x^H)$ and define $\hat{\overline{T}}_{j,\epsilon}(x^H)$, $\hat{\underline{t}}_j(x^H)$, $\hat{\underline{T}}_{j,\epsilon}(x^H)$, $\hat{j}(t, \epsilon; x^H)$, and $\hat{S}(x^H)$ analogously to (60)–(64).

By the reasoning up to (66), given any allocation y^H satisfying (48) and (52), there is a corresponding $\tilde{y}^H$ (and $\tilde{x}^H$) also satisfying (48) and (52) but for which we also have

$$\hat{S}(\tilde{x}^H) = \emptyset, \tag{90}$$

such that H is indifferent between y^H and $\tilde{y}^H$ given L's best responses.

Consider a feasible allocation y^H (and corresponding x^H) satisfying (48), (52), and (90) but not (54). Given a choice of interval-length ϵ, consider the allocation $\tilde{y}^H(\epsilon)$ (and corresponding $\tilde{x}^H(\epsilon)$) with

$$\tilde{y}_t^H(\epsilon) = \begin{cases} y_t^H, & t \notin \hat{\overline{T}}(x^H) \cup \hat{\underline{T}}(x^H); \\ 0, & t \in \hat{\underline{T}}(x^H); \\ \Lambda^H(x^H) e^{\alpha^L (\hat{\underline{t}}_{\hat{j}(t,\epsilon;x^H)-\epsilon}(x^H) - z^* - \hat{j}(t,\epsilon;x^H)) - \alpha^H \underline{t}_{\hat{j}(t,\epsilon;x^H)}(x^H)}, & t \in \hat{\overline{T}}(x^H). \end{cases}$$

Note that $\tilde{x}^H(\epsilon)$ satisfies (56) for all ϵ.

To demonstrate that $\tilde{y}^H$ is feasible, let us show that its allocation to each $\hat{\overline{T}}_{j,\epsilon}(x^H)$ is weakly (and indeed strictly) less than y^H's allocation to the corresponding $\hat{\underline{T}}_{j,\epsilon}(x^H)$. From the analogs to (61) and (62), we have the analog to (67). Observe that $t \geq \hat{\underline{t}}_{\hat{j}(t,\epsilon;x^H)-\epsilon}(x^H)\ \forall t \in \hat{\underline{T}}_{j,\epsilon}(x^H)$ and $t < z^* + j\ \forall t \in \hat{\overline{T}}_{j,\epsilon}(x^H)$. Also,

$$y_t^H \geq \Lambda^H(x^H)e^{-\alpha^H \hat{\underline{t}}_j(x^H)} \quad \forall t \in \hat{\underline{T}}_{j,\epsilon}(x^H).$$

Thus

$$\begin{aligned}
\int_{\hat{\overline{T}}_{j,\epsilon}(x^H)} e^{-\alpha^L(z^*+j)}dt &\leq \int_{\hat{\underline{T}}_{j,\epsilon}(x^H)} e^{-\alpha^L \hat{\underline{t}}_j(x^H)}dt \\
\implies \int_{\hat{\overline{T}}_{j,\epsilon}(x^H)} \Lambda^H(x^H)e^{\alpha^L(\hat{\underline{t}}_{j-\epsilon}(x^H)-z^*-j)-\alpha^H \hat{\underline{t}}_j(x^H)}dt &\leq \int_{\hat{\underline{T}}_{j,\epsilon}(x^H)} \Lambda^H(x^H)e^{-\alpha^H \hat{\underline{t}}_j(x^H)}dt \\
&< \int_{\hat{\underline{T}}_{j,\epsilon}(x^H)} \Lambda^H(x^H)e^{-\alpha^H t}dt.
\end{aligned}$$

Summing across $j \in \epsilon\mathbb{N}$, it follows that, since y^H is feasible, $\tilde{y}^H$ is also feasible.

By calculations precisely analogous to those from (69) to (79)—here simply moving H's spending forward from $\hat{\underline{T}}$ to $\hat{\overline{T}}$, rather than backward from $\overline{T}$ to $\underline{T}$—the total net utility gain for H from the shift from y^H to $\tilde{y}^H(\epsilon)$ converges, as $\epsilon \to 0$, to a value strictly greater than

$$\frac{(\Lambda^L(x^H))^{1-\gamma}}{1-\gamma} \int_{\hat{\overline{T}}(x^H)} \left(\left(\frac{\Lambda^H(x^H)}{\Lambda^L(x^H)} e^{(\alpha^L-\alpha^H)t} \right)^{1-\gamma} - 1 \right) \left(e^{(\delta^L-\delta^H)\hat{\underline{t}}_{t-z^*}(x^H)} - e^{(\delta^L-\delta^H)t} \right) dt.$$

This is positive, like its analog (79): both the coefficient outside the integral and the first factor in the integral have changed sign. Therefore the total net utility gain is positive; H strictly prefers $\tilde{y}^H(\epsilon)$, for sufficiently small ϵ, to y^H given L's best responses.

We have shown that, if $\gamma < 1$, for any feasible allocation y^H with x^H satisfying (48), (52), and (66), but not (54), there is a feasible $\tilde{y}^H$ (and corresponding $\tilde{x}^H$) which H strictly prefers, given L's best responses, with threshold times $\underline{\tilde{t}}(\tilde{x}^H), \overline{\tilde{t}}(\tilde{x}^H)$ such that

$$x_t^L(x^H) = 0 \;\; \forall t \in [\underline{\tilde{t}}(\tilde{x}^H), \overline{\tilde{t}}(\tilde{x}^H)), \quad x_t^H = 0 \;\; \forall t \notin [\underline{\tilde{t}}(\tilde{x}^H), \overline{\tilde{t}}(\tilde{x}^H)). \tag{91}$$

(In the case of the $\tilde{x}^H$ constructed just above, $\underline{\tilde{t}}(\tilde{x}^H) = z^*$ and $\overline{\tilde{t}}(\tilde{x}^H) = \overline{q}(x^H)$.)

Consider a feasible allocation y^H (and corresponding x^H) satisfying (91) for some $\underline{\tilde{t}}(x^H), \overline{\tilde{t}}(x^H)$. Because L, in equilibrium, allocates his resources δ^L-optimally across

$[0, \underline{\tilde{t}}(x^H)) \cup [\bar{\bar{\tilde{t}}}(x^H), \infty)$, we have

$$x_t^L(x^H) = \begin{cases} 0, & t \in [\underline{\tilde{t}}(x^H), \bar{\bar{\tilde{t}}}(x^H)); \\ \Lambda^L(x^H) e^{(r-\alpha^L)t}, & t \in [0, \underline{\tilde{t}}(x^H)) \cup [\bar{\bar{\tilde{t}}}(x^H), \infty), \end{cases} \tag{92}$$

up to measure-zero deviations, where $\Lambda^L(x^H)$ satisfies

$$B^L = \int_0^{\underline{\tilde{t}}(x^H)} \Lambda^L(x^H) e^{-\alpha^L t} dt + \int_{\bar{\bar{\tilde{t}}}(x^H)}^{\infty} \Lambda^L(x^H) e^{-\alpha^L t} dt$$

$$\implies \quad \Lambda^L(x^H) = \frac{B^L \alpha^L}{1 + e^{-\alpha^L \bar{\bar{\tilde{t}}}(x^L)} - e^{-\alpha^L \underline{\tilde{t}}(x^L)}}. \tag{93}$$

Likewise, fixing $\underline{\tilde{t}}$ and $\bar{\bar{\tilde{t}}}$, denote H's most preferred x^H satisfying (92) with $\underline{\tilde{t}}(x^H) = \underline{\tilde{t}}$ and $\bar{\bar{\tilde{t}}}(x^H) = \bar{\bar{\tilde{t}}}$ by $x^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}]$. $x^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}]$ must allocate H's resources δ^H-optimally across $[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}})$. Spending δ^H-optimally for an arbitrarily long interval will not be compatible with (52), and thus not with (91), in equilibrium; L will eventually prefer spending during the interval. Nevertheless, we will find the $\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}$ that would be optimal for H if H *could* spend δ^H-optimally across an arbitrary $[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}})$, instead of eventually having to switch to a δ^L-optimal schedule (as in (50)). We will then see that the δ^H-optimal schedule satisfying (91) is indeed compatible with (52) and thus (91) in equilibrium.

So we have

$$x_t^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}] = \begin{cases} \Lambda^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}] e^{(r-\alpha^H)t}, & t \in [\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}); \\ 0, & t \in [0, \underline{\tilde{t}}) \cup [\bar{\bar{\tilde{t}}}, \infty), \end{cases}$$

where $\Lambda^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}]$ satisfies

$$\int_{\underline{\tilde{t}}}^{\bar{\bar{\tilde{t}}}} \Lambda^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}] e^{-\alpha^H t} dt = B^H \quad \implies \quad \Lambda^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}] = \frac{B^H \alpha^H}{e^{-\alpha^H \underline{\tilde{t}}} - e^{-\alpha^H \bar{\bar{\tilde{t}}}}}. \tag{94}$$

$(x^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}], x^L(x^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}]))$ delivers H payoff

$$\Bigg[\int_0^{\underline{\tilde{t}}} e^{-\delta^H t} \Big(\Lambda^L(x^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}]) e^{(r-\alpha^L)t} \Big)^{1-\gamma} dt + \int_{\underline{\tilde{t}}}^{\bar{\bar{\tilde{t}}}} e^{-\delta^H t} \Big(\Lambda^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}] e^{(r-\alpha^H)t} \Big)^{1-\gamma}$$

$$+ \int_{\bar{\bar{\tilde{t}}}}^{\infty} e^{-\delta^H t} \Big(\Lambda^L(x^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}]) e^{(r-\alpha^L)t} \Big)^{1-\gamma} dt \Bigg] \frac{1}{1-\gamma}.$$

Rearranging (94) yields

$$\bar{\bar{\tilde{t}}} = -\ln \Big(e^{-\alpha^H \underline{\tilde{t}}} - \frac{B^H \alpha^H}{\Lambda^H} \Big) / \alpha^H. \tag{95}$$

Substitute (93) for $\Lambda^L(x^H[\underline{\tilde{t}}, \bar{\bar{\tilde{t}}}])$ and then (95) for $\bar{\bar{\tilde{t}}}$, so as to reframe H's optimization problem as being over choices of $\underline{\tilde{t}}$ and Λ^H (with $\bar{\bar{\tilde{t}}}$ endogenous) instead of $\underline{\tilde{t}}$ and $\bar{\bar{\tilde{t}}}$ (with Λ^H endogenous). Integrating and simplifying yields

$$\left[\left(\frac{B^L\alpha^L}{1+\left(e^{-\alpha^H\underline{\tilde{t}}}-\frac{B^H\alpha^H}{\Lambda^H}\right)^{\frac{\alpha^L}{\alpha^H}}-e^{-\alpha^L\underline{\tilde{t}}}}\right)^{1-\gamma}\frac{1+\left(e^{-\alpha^H\underline{\tilde{t}}}-\frac{B^H\alpha^H}{\Lambda^H}\right)^{\frac{\alpha^L+\delta^H-\delta^L}{\alpha^H}}-e^{-(\alpha^L+\delta^H-\delta^L)\underline{\tilde{t}}}}{\alpha^L+\delta^H-\delta^L} + (\Lambda^H)^{-\gamma}B^H\right]\frac{1}{1-\gamma}. \tag{96}$$

Differentiating with respect to $\underline{\tilde{t}}$ (and then re-introducing Λ^L and $\bar{\bar{\tilde{t}}}$ in places, for clarity) gives

$$-\frac{1}{\alpha^L+\delta^H-\delta^L}\frac{(\Lambda^L)^{2-\gamma}}{B^L\alpha^L}\left(1+e^{-(\alpha^L+\delta^H-\delta^L)\bar{\bar{\tilde{t}}}}-e^{-(\alpha^L+\delta^H-\delta^L)\underline{\tilde{t}}}\right)$$
$$+\frac{(\Lambda^L)^{1-\gamma}}{1-\gamma}\left(e^{-(\alpha^L+\delta^H-\delta^L)\underline{\tilde{t}}}-\left(e^{-\alpha^H\underline{\tilde{t}}}-\frac{B^H\alpha^H}{\Lambda^H}\right)^{\frac{\alpha^H-\alpha^L}{\alpha^H}(\gamma-1)}e^{-\alpha^H\underline{\tilde{t}}}\right).$$

The first of these two added expressions is negative. The second is also negative, as can be seen from the fact that it is zero when the B^H term explicitly represented equals zero, and decreases as this term increases.

Thus, for any feasible x^H satisfying (48), (52), (66), and (91) but not

$$\underline{\tilde{t}}(x^H) = 0,$$

there is a strictly preferred feasible $\tilde{x}^H$ (with, incidentally, $\Lambda^H(\tilde{x}^H) = \Lambda^H(x^H)$) satisfying all five conditions.

H's favorite strategy in this class is derived for all $\gamma \neq 1$ in (80)–(85). Denote it by x^{H*}, as done there, with $t^* \equiv \bar{\bar{\tilde{t}}}(x^{H*})$ in the $\gamma < 1$ case.

Last steps

Letting $x^{L*} \equiv x^L(x^{H*})$ and

$$M \equiv 1 + \frac{B^H\alpha^H}{B^L\alpha^L}\,\eta,$$

it follows from (80)–(83) and (84) in the $\gamma \neq 1$ cases, and (89) in the $\gamma = 1$ case, that

$$x_t^{H*} = \begin{cases} B^H\alpha^H\frac{M}{M-1}e^{(r-\alpha^H)t}, & t < t^*; \\ 0, & t \geq t^* \end{cases}$$

and

$$x_t^{L*} = \begin{cases} 0, & t < t^*; \\ B^L \alpha^L M^{\frac{\alpha^L}{\alpha^H}} e^{(r-\alpha^L)t}, & t \geq t^* \end{cases}$$

for all $\gamma > 0$, where

$$t^* \equiv \ln(M)/\alpha^H.$$

Given L's strategy $x^L(\cdot)$ (or any measure-zero deviation from it), which is L's unique optimal strategy, H strictly prefers x^{H*} to any alternative spending schedule $x^{H(0)}$ that is a positive-measure deviation from x^{H*} (a "PMD"). This follows from the constructive derivation of x^{H*}, as summarized:

- For any PMD $x^{H(0)}$, we can construct a PMD $x^{H(1)}$ satisfying (48) such that $x^{H(1)} \sim^H x^{H(0)}$.
- For any PMD $x^{H(1)}$ satisfying (48):
  - If $x^{H(1)}$ does not satisfy (49) almost everywhere, we can construct an $x^{H(1')}$ satisfying (48) and (49) such that $x^{H(1')} \succ^H x^{H(1)}$. $x^{H*} \succ^H x^{H(1')}$, by the full derivation above, so $x^{H*} \succ^H x^{H(1)} \sim^H x^{H(0)}$.
  - If $x^{H(1)}$ does satisfy (49) almost everywhere, we can construct a PMD $x^{H(2)}$ satisfying (48) and (49) such that $x^{H(2)} \sim^H x^{H(1)}$.
- For any PMD $x^{H(2)}$ satisfying (48) and (49), we can construct a PMD $X^{H(3)}$ satisfying (48) and (52) such that $x^{H(3)} \sim^H x^{H(2)}$.
- For any PMD $x^{H(3)}$ satisfying (48) and (52), we can construct a PMD $x^{H(4)}$ satisfying (48), (52), and (66) such that $x^{H(4)} \sim^H x^{H(3)}$.
- For any PMD $x^{H(4)}$ satisfying (48), (52), and (66), we can construct an $x^{H(5)}$ satisfying (48), (52), (66), and (56) ($\gamma \geq 1$) or (91) ($\gamma < 1$) such that $x^{H(5)} \succsim^H x^{H(4)}$ and $x^{H(5)} \neq x^{H*}$.
- H strictly prefers x^{H*} to all other spending schedules satisfying (48), (52), (66), and (56) ($\gamma \geq 1$) or (91) ($\gamma < 1$). So $x^{H*} \succ^H x^{H(5)} \succsim^H x^{H(0)}$.

This completes the result.

A.5 Proof of Proposition 5

Open-loop and Stackelberg schedules

Before beginning, it will be useful to introduce some notation regarding "open-loop" and "Stackelberg" schedules in general, and familiarize ourselves with some of their properties.

Open-loop schedules

Given a nonempty interval $[t, \bar{t})$ and a pair of budgets B^H, B^L to be allocated across it at least one of which is positive, let $x^o[t, \bar{t}, B^L, B^H]$—suppressing some or all of these arguments when clear—denote the (potentially truncated) schedule on which B^H is allocated δ^H-optimally across $[t, t^o)$ and B^L is allocated δ^L-optimally across $[t^o, T)$, where t^o (or $t^o[t, \bar{t}, B^L, B^H]$, etc.) is the point in $[t, \bar{t}]$ that uniquely renders collective spending continuous at t^o. Call this schedule the "open-loop" schedule, and call t^o its "regime-change point".

$$x_s^{H(o)} = \begin{cases} \frac{B^H \alpha^H}{1 - e^{-\alpha^H (t^o - t)}} e^{(r - \alpha^H)(s-t)}, & s \in [t, t^o); \\ 0 & s \in [t^o, \bar{t}), \end{cases} \tag{97}$$

$$x_s^{L(o)} = \begin{cases} 0, & s \in [t, t^o); \\ \frac{B^L \alpha^L}{e^{-\alpha^L (t^o - t)} - e^{-\alpha^L (\bar{t} - t)}} e^{(r - \alpha^L)(s-t)}, & s \in [t^o, \bar{t}). \end{cases}$$

These formulas for x^o, and a proof that it is the unique equilibrium schedule across its interval when the schedule from $\bar{t}$ onward depends only on $\{B^i_{\bar{t}}\}$, follow immediately from (17) and the proof of Proposition 3 (Appendix A.3).

$t^o[B^H, B^L]$ is $\mathcal{C}^1$ in its arguments, by the implicit function theorem and the fact that t^o uniquely satisfies

$$\lim_{s \to t^{o-}} x_s^{H(o)} - x_{t^o}^{L(o)} = 0,$$

whose left-hand side is $\mathcal{C}^1$ in t^o and $\{B^i\}$. Furthermore, since the first term of this difference increases in B^H and decreases in t^o and the second increases in B^L but decreases in t^o, $t^o[B^H, B^L]$ is increasing (decreasing) in B^H (B^L), as is intuitive.

It follows that, for all $s \in [t, \bar{t})$, the open-loop collective spending rate $X_s^o[B^H, B^L]$ strictly increases in B^i for both i. Since from (97) X_s^o is $\mathcal{C}^1$ in t^o and in $\{B^i\}$, and since we found that t^o is itself $\mathcal{C}^1$ in $\{B^i\}$, X_s^o is $\mathcal{C}^1$ in $\{B^i\}$ with respect to its total derivatives (i.e. accounting for the effects of changes in B^i on t^o). So, for each i, $U^i(x^o[B^i, B^{-i}])$ is $\mathcal{C}^1$ in B^i and B^{-i}.

Stackelberg schedules

Given a nonempty interval $[t, T)$ and a pair of budgets B^H, B^L to be allocated across it at least one of which is positive, call a truncated schedule across the interval <u>IC-polarized</u> if there is a "regime-change point" $\tilde{t} \in [t, T]$ such that H is the only spender across $[t, \tilde{t})$ and L allocates B^L δ^L-optimally across $[\tilde{t}, T)$. We will use T rather than $\bar{t}$ to denote the upper end of the interval because we will only consider intervals ending at the end of the game. The "IC" stands for "incentive-compatible", and refers to the fact that if H exhausts her budget at $\tilde{t}$ and L is restricted from spending before $\tilde{t}$, L can have no incentive to do anything other than allocate $B^L_{\tilde{t}}$ δ^L-optimally across $[\tilde{t}, T)$.

Call H's favorite feasible IC-polarized schedule across the interval the "Stackelberg schedule". Denote it by $x^*[t, T, B^H, B^L]$, and its regime-change point by $t^*[t, T, B^H, B^L]$. As with x^o and t^o, we will suppress some or all of these arguments when clear. We will now show that for any t, $T > t$, $B^H > 0$, and $B^L > 0$, $x^*[t, T, B^H, B^L]$ exists and is unique; existence and uniqueness when one budget or both is zero is trivial. We will then note some of its properties.

(The Stackelberg schedule gets its name from the fact that, as we will later see, $x^*[0, \infty, B^H, B^L]$ is the Stackelberg schedule found in Proposition 4. We will not consider whether, on a finite horizon, x^* is the unique equilibrium schedule of a finite-horizon transformation of the Stackelberg game.)

Let $\tilde{t}$ denote a candidate regime-change point.

By the logic of the proof of Proposition 1, the δ^L-optimal allocation of B_t^L across $[\tilde{t}, T)$ equals

$$x_s^L = \frac{B_t^L \alpha^L}{e^{-\alpha^L(\tilde{t}-t)} - e^{-\alpha^L(T-t)}} e^{(r-\alpha^L)(s-t)}, \quad s \in [\tilde{t}, T). \tag{98}$$

Likewise, the δ^H-optimal allocation of B_t^H across $[t, \tilde{t})$—which, if $t^* = \tilde{t}$, must of course be the allocation adopted under x^*—equals

$$x_t^H = \frac{B_t^H \alpha^H}{1 - e^{-\alpha^H(\tilde{t}-t)}} e^{(r-\alpha^H)(s-t)}, \quad s \in [t, \tilde{t}). \tag{99}$$

Given that L spends nothing until $\tilde{t}$ and then adopts (98), and that H adopts (99), H's favorite location for $\tilde{t}$ is that which maximizes δ^H-discounted utility:

$$\begin{aligned} t^* = \operatorname*{argmax}_{\tilde{t}} \Bigg[& \int_t^{\tilde{t}} e^{-\delta^H(s-t)} u\Big(\frac{B_t^H \alpha^H}{1 - e^{-\alpha^H(\tilde{t}-t)}} e^{(r-\alpha^H)(s-t)}\Big) ds \\ & + \int_{\tilde{t}}^T e^{-\delta^H(s-t)} u\Big(\frac{B_t^L \alpha^L}{e^{-\alpha^L(\tilde{t}-t)} - e^{-\alpha^L(T-t)}} e^{(r-\alpha^L)(s-t)}\Big) ds \Bigg]. \end{aligned} \tag{100}$$

We will not find a closed-form expression for t^*, but we will show that it exists and is unique.

Integrating (100) yields

$$\frac{1}{1-\gamma}\Bigg[\frac{1}{\alpha^H}\Big(1-e^{-\alpha^H(\tilde{t}-t)}\Big)^{\gamma}\big(B_t^H\alpha^H\big)^{1-\gamma} \tag{101}$$

$$+\frac{1}{\delta^H+\alpha^L-\delta^L}e^{-\gamma\alpha^H(\tilde{t}-t)}\Big(1-e^{(\delta^L-\delta^H-\alpha^L)(T-\tilde{t})}\Big)\Big(\frac{B_t^L\alpha^L}{1-e^{-\alpha^L(T-\tilde{t})}}\Big)^{1-\gamma}\Bigg],\quad \gamma\neq 1;$$

$$\frac{1}{\delta^H}\Bigg[\Bigg(\ln\Big(\frac{B_t^H\delta^H}{1-e^{-\delta^H(\tilde{t}-t)}}\Big)+\frac{r}{\delta^H}-1\Bigg)\Big(1-e^{-\delta^H(\tilde{t}-t)}\Big)$$

$$+\Bigg(\ln\Big(\frac{B_t^L\alpha^L}{e^{-\delta^L(\tilde{t}-t)}-e^{-\delta^L(T-t)}}\Big)+(r-\delta^L)\Big(\frac{1}{\delta^H}-t\Big)\Bigg)\Big(e^{-\delta^H(\tilde{t}-t)}-e^{-\delta^H(T-t)}\Big)$$

$$+(\delta^H-\delta^L)e^{-\delta^H(\tilde{t}-t)}\tilde{t}-(r-\delta^L)e^{-\delta^H(T-t)}T\Bigg],\quad \gamma=1.$$

Differentiating with respect to $\tilde{t}$ then yields

$$\Big(B_t^H\alpha^H\Big)^{1-\gamma}\frac{\gamma}{1-\gamma}\big(1-e^{-\alpha^H(\tilde{t}-t)}\big)^{\gamma-1}e^{-\alpha^H(\tilde{t}-t)} \tag{102}$$

$$+\Big(B_t^L\alpha^L\Big)^{1-\gamma}\Big(1-e^{-\alpha^L(T-\tilde{t})}\Big)^{\gamma-2}e^{-\gamma\alpha^H(\tilde{t}-t)}$$

$$\Bigg(\frac{1}{1-\gamma}\Big(e^{-\alpha^L(T-\tilde{t})}-1\Big)+\frac{\alpha^L}{\delta^H+\alpha^L-\delta^L}\Big(1-e^{(\delta^L-\delta^H-\alpha^L)(T-\tilde{t})}\Big)\Bigg),\quad \gamma\neq 1;$$

$$e^{-\delta^H(\tilde{t}-t)}\Bigg(\ln\Big(\frac{B_t^H\delta^H}{1-e^{-\delta^H(\tilde{t}-t)}}\Big)-\ln\Big(\frac{B_t^L\delta^L}{1-e^{-\delta^L(T-\tilde{t})}}\Big)$$

$$+\frac{\delta^L}{1-e^{-\delta^L(T-\tilde{t})}}\,\frac{1-e^{-\delta^H(T-\tilde{t})}}{\delta^H}\Bigg),\qquad \gamma=1,$$

which is continuous in $\tilde{t}$ and approaches a positive value as $\tilde{t}\to t^+$ and a negative value as $\tilde{t}\to T^-$. This guarantees at least one interior solution. Though the second derivative (i.e. the derivative of (102)) is not always negative, it can be proven as follows that a unique $\tilde{t}=t^*$ sets (102) equal to zero.

Rearranging (102) yields

$$\text{Ⓐ}_{\tilde{t}}\big(\text{Ⓑ}_{\tilde{t}}\text{Ⓒ}_{\tilde{t}}+(1-\text{Ⓑ}_{\tilde{t}})\text{Ⓓ}_{\tilde{t}}-1\big), \tag{103}$$

where

$$\textcircled{A}_{\tilde{t}} \equiv \frac{B_t^H \alpha^H}{e^{\alpha^H(\tilde{t}-t)}-1}\left(\frac{B_t^H \alpha^H}{1-e^{-\alpha^H(\tilde{t}-t)}}\right)^{-\gamma},$$

$$\textcircled{B}_{\tilde{t}} \equiv \frac{B_t^L \alpha^L \big/ \left(1-e^{-\alpha^L(T-\tilde{t})}\right)}{B_t^H \alpha^H \big/ \left(e^{\alpha^H(\tilde{t}-t)}-1\right)},$$

$$\textcircled{C}_{\tilde{t}} \equiv \left(B_t^L\right)^{-\gamma}\left(\alpha^L\right)^{1-\gamma}\frac{1-e^{(\delta^L-\alpha^L-\delta^H)(T-\tilde{t})}}{\delta^H+\alpha^L-\delta^L}\left(1-e^{-\alpha^L(T-\tilde{t})}\right)^{\gamma-1}\cdot\left(\frac{B_t^H \alpha^H}{e^{\alpha^H(\tilde{t}-t)}-1}\right)^{\gamma},$$

$$\textcircled{D}_{\tilde{t}} \equiv \frac{1}{1-\gamma}\frac{\left(\frac{B_t^H\alpha^H}{e^{\alpha^H(\tilde{t}-t)}-1}\right)^{1-\gamma}-\left(\frac{B_t^L\alpha^L}{1-e^{-\alpha^L(T-\tilde{t})}}\right)^{1-\gamma}}{\frac{B_t^H\alpha^H}{e^{\alpha^H(\tilde{t}-t)}-1}-\frac{B_t^L\alpha^L}{1-e^{-\alpha^L(T-\tilde{t})}}}\cdot\left(\frac{B_t^H \alpha^H}{e^{\alpha^H(\tilde{t}-t)}-1}\right)^{\gamma}, \quad \gamma \neq 1;$$

$$\frac{\ln\left(\frac{B_t^H\delta^H}{e^{\delta^H(\tilde{t}-t)}-1}\right)-\ln\left(\frac{B_t^L\delta^L}{1-e^{-\delta^L(T-\tilde{t})}}\right)}{\frac{B_t^H\delta^H}{e^{\delta^H(\tilde{t}-t)}-1}-\frac{B_t^L\delta^L}{1-e^{-\delta^L(T-\tilde{t})}}}\cdot\frac{B_t^H\delta^H}{e^{\delta^H(\tilde{t}-t)}-1}, \qquad \gamma = 1.$$

$\textcircled{A}_{\tilde{t}}$ is positive for all $\tilde{t} > t$. To find $\tilde{t}$ that set (103) equal to zero, therefore, we need only focus on the expression within the outer parentheses.

We will first show that (103) is negative if $\textcircled{B}_{\tilde{t}} \geq 1$. This amounts to the intuitive observation that, if H has set $\tilde{t}$ so high that spending weakly jumps *up* at $\tilde{t}$, because her resources from t to $\tilde{t}$ are spread so thinly and L's resources from $\tilde{t}$ to T are so concentrated, then H does not want to raise $\tilde{t}$ further.

When $\textcircled{B}_{\tilde{t}} \geq 1$, $\textcircled{D}_{\tilde{t}}$ (or its limit in the $\textcircled{B}_{\tilde{t}} = 1$ case) is weakly less than 1. This can be seen from the facts that $u(\cdot)$ is convex, that the first term of $\textcircled{D}_{\tilde{t}}$ is the slope of the line segment connecting the points

$$\left(\frac{B_t^H\alpha^H}{e^{\alpha^H(\tilde{t}-t)}-1}, u\left(\frac{B_t^H\alpha^H}{e^{\alpha^H(\tilde{t}-t)}-1}\right)\right), \quad \left(\frac{B_t^L\alpha^L}{1-e^{-\alpha^L(T-\tilde{t})}}, u\left(\frac{B_t^L\alpha^L}{1-e^{-\alpha^L(T-\tilde{t})}}\right)\right) \tag{104}$$

(or the slope at that point when they are equal), and the second term of $\textcircled{D}_{\tilde{t}}$ divides the first by the derivative at the lower of the two points. When $\textcircled{B}_{\tilde{t}} \geq 1$, (103) is thus weakly less than

$$\textcircled{A}_{\tilde{t}}\,\textcircled{B}_{\tilde{t}}\left(\textcircled{C}_{\tilde{t}} - \textcircled{D}_{\tilde{t}}\right).$$

But

$$\textcircled{D}_{\tilde{t}} > \left(\frac{B_t^L\alpha^L}{1-e^{-\alpha^L(T-\tilde{t})}}\right)^{-\gamma}\left(\frac{B_t^H\alpha^H}{e^{\alpha^H(\tilde{t}-t)}-1}\right)^{\gamma}$$
$$= \textcircled{C}_{\tilde{t}}\,\frac{\alpha^L+\delta^H-\delta^L}{\alpha^L}\,\frac{1-e^{-\alpha^L(T-\tilde{t})}}{1-e^{-(\alpha^L+\delta^H-\delta^L)(T-\tilde{t})}} > \textcircled{C}_{\tilde{t}}.$$

So (103) is negative.

Since $\textcircled{B}_{\tilde{t}}$ is strictly increasing in $\tilde{t}$ from 0 at $\tilde{t} = t$ to ∞ at $\tilde{t} = T$, there is a unique $\bar{\tilde{t}}$ such that $\textcircled{B}_{\bar{\tilde{t}}} = 1$, and any $\tilde{t}$ setting (103) equal to 0 must be less than $\bar{\tilde{t}}$. Denote some such $\tilde{t}$ by t^*. We will now show that it is unique in $(t, \bar{\tilde{t}})$.

We will do this by showing that, if $\textcircled{B}_{t^*}\textcircled{C}_{t^*} + (1 - \textcircled{B}_{t^*})\textcircled{D}_{t^*} = 1$, this expression is strictly lower for $\tilde{t} \in (t^*, \bar{\tilde{t}})$. This will prove that the $\tilde{t}$ at which this sum equals 1, and thus at which (103) equals 0, is unique.

Fixing $t^* \in (t, \bar{\tilde{t}})$ at which (103)=0, choose $\tilde{t} \in (t^*, \bar{\tilde{t}})$.

$$\begin{aligned}&\textcircled{B}_{t^*}\textcircled{C}_{t^*} + (1 - \textcircled{B}_{t^*})\textcircled{D}_{t^*} - \textcircled{B}_{\tilde{t}}\textcircled{C}_{\tilde{t}} - (1 - \textcircled{B}_{\tilde{t}})\textcircled{D}_{\tilde{t}} \\ &= \textcircled{B}_{\tilde{t}}\big(\textcircled{C}_{t^*} - \textcircled{C}_{\tilde{t}}\big) + \big(\textcircled{D}_{t^*} - \textcircled{C}_{t^*}\big)\big(\textcircled{B}_{\tilde{t}} - \textcircled{B}_{t^*}\big) + \big(1 - \textcircled{B}_{\tilde{t}}\big)\big(\textcircled{D}_{t^*} - \textcircled{D}_{\tilde{t}}\big). \qquad (105)\end{aligned}$$

We will show that each of these terms is positive.

$\textcircled{B}_{\tilde{t}}$ and $1 - \textcircled{B}_{\tilde{t}}$ are positive for all $\tilde{t} \in (t, \bar{\tilde{t}})$, by definition of $\bar{\tilde{t}}$. Because $\textcircled{B}_{\tilde{t}}$ increases in $\tilde{t}$, $\textcircled{B}_{\tilde{t}} - \textcircled{B}_{t^*}$ is also positive.

Putting aside the positive constant $(B_t^H \alpha^H)^\gamma (B_t^L)^{-\gamma} (\alpha^L)^{1-\gamma} / (\delta^H + \alpha^L - \delta^L)$, $\textcircled{C}_{\tilde{t}}$ can be split into three positive terms each of which decreases in $\tilde{t}$:

$$\frac{1 - e^{(\delta^L - \alpha^L - \delta^H)(T - \tilde{t})}}{1 - e^{-\alpha^L (T - \tilde{t})}} \;\cdot\; \big(1 - e^{-\alpha^L (T - \tilde{t})}\big)^\gamma \;\cdot\; \big(e^{\alpha^H (\tilde{t} - t)} - 1\big)^{-\gamma}.$$

$\textcircled{C}_{t^*} - \textcircled{C}_{\tilde{t}}$ is thus positive.

To see that $\textcircled{D}_{t^*} - \textcircled{C}_{t^*}$ is positive, observe first that, for $t^* < \bar{\tilde{t}}$, $\textcircled{D}_{t^*} > 1$. Its first term is the slope of the line segment connecting points (104) on the graph of $u(\cdot)$ and the second divides it by the derivative of $u(\cdot)$ at what is, given $t^* < \bar{\tilde{t}}$ and thus $\textcircled{B}_{t^*} < 1$, the higher of the two points. Setting (103)=0 and rearranging, we then have

$$\begin{aligned}\textcircled{B}_{t^*}\textcircled{C}_{t^*} + (1 - \textcircled{B}_{t^*})\textcircled{D}_{t^*} &= 1 \qquad (106)\\ \implies \textcircled{C}_{t^*} &= \textcircled{D}_{t^*} - \frac{\textcircled{D}_{t^*} - 1}{\textcircled{B}_{t^*}} \\ &< \textcircled{D}_{t^*}.\end{aligned}$$

To see that $\textcircled{D}_{t^*} - \textcircled{D}_{\tilde{t}}$ is positive, recall once again that $\textcircled{D}_{\tilde{t}}$ is the ratio between the slope of the line segment connecting points (104) and the derivative at the higher point. Labeling the points $(p_1, u(p_1))$ and $(p_2, u(p_2))$ respectively, if both p_1 and p_2 were multiplied by some $m < 1$, both the slope of the line segment and the derivative at the higher point would be multiplied by $m^{-\gamma}$, and the ratio would be unchanged. In the move from $\textcircled{D}_{t^*}$ to $\textcircled{D}_{\tilde{t}}$, however, because

$$\frac{\partial}{\partial \tilde{t}} \frac{B_t^H \alpha^H}{e^{\alpha^H (\tilde{t} - t)} - 1} < 0 \;\text{ but }\; \frac{\partial}{\partial \tilde{t}} \frac{B_t^L \alpha^L}{1 - e^{-\alpha^L (T - \tilde{t})}} > 0, \qquad (107)$$

p_1 is indeed multiplied by some $m < 1$ and the derivative at the higher point multiplied by $m^{-\gamma}$, but p_2 is multiplied by $\overline{m} > 1 > m$. The slope of the line segment is multiplied by less than $m^{-\gamma}$, so the ratio falls.

We have now found that H's favorite feasible IC-polarized schedule x^* is unique. We have also found that $\text{Ⓑ}_{t^*} > 1$. That is, if H allocates B_t^H δ^H-optimally across $[t, t^*)$ and L allocates B_t^L δ^L-optimally across $[t^*, T)$, then collective spending weakly falls at t^*. This establishes that $t^* < t^o$.

Precisely as with t^o in the case of the open-loop schedule, we can use the implicit function theorem to establish that $t^*[B^H, B^L]$ is $\mathcal{C}^1$ in both arguments, increasing in B^H and decreasing in B^L. Increases to B^H increase $X_s^*[B^H, B^L]$ for all $s \in [t, T)$, as with $X^o[\cdot]$, though this is no longer obvious: it is proven in the "Final point: Unique equilibrium schedule given placement of first announcement" section below. Now, however, increases to B^L do not necessarily do so.

Overview

First, we will show that the unique equilibrium schedule of the game with finite horizon T resembles the Stackelberg schedule of Proposition 4, x^*, but on a finite horizon. We will also show that this T-horizon schedule converges pointwise almost everywhere to x^* as $T \to \infty$. This will nearly prove part (a), as it will prove that if x^* is an equilibrium schedule, it is the unique limit equilibrium schedule.

Second, we will prove that for any polarized equilibrium σ^* of the infinite-horizon game, $x(\sigma^*) = x^*$. This will nearly prove part (b), as it will prove that if a polarized equilibrium schedule exists, it is unique and equals x^*.

Finally, we will show that a polarized equilibrium (of the infinite-horizon game) exists. This will conclude the proofs of (a) and (b).

The proof of the unique equilibrium schedule of the game with finite horizon T proceeds by backward induction. Let $\overline{\tau} \equiv \max\{G^{(n)} \cap [0, T)\}$ denote the highest grid point in T-horizon game n. We will therefore begin by finding the unique equilibrium schedule of each subgame beginning at a node $h_{|\overline{\tau}}$.

If $B^i(h_{|\overline{\tau}}) = 0$ for either i, the subgame equilibrium schedule is trivial: any player $-i$ with a positive budget allocates it δ^{-i}-optimally from $\overline{\tau}$ to T. Announcements are irrelevant.

If $B^i(h_{|\overline{\tau}}) > 0$ for both i, the proof of the unique subgame equilibrium schedule following $h_{|\overline{\tau}}$ will itself proceed by backward induction. We will find equilibrium schedules

1. after the second announced node, if any;

2. after the second announcement has been made, if any;

3. after the first announced node (i.e. we will find when the second announcement may occur in equilibrium), if any;

4. after the first announcement has been made, if any; and then

5. from $\overline{\tau}$ before any announcements have been made (i.e. we will find when the first announcement may occur in equilibrium).

Let σ^* be an equilibrium.

Final period

Unique equilibrium schedule after second announced node

The schedule implemented by σ^* following the second announcement after $\overline{\tau}$ follows immediately from (17) and the proof of Proposition 3 (Appendix A.3). Given a type 5 node $h_{|t}$, $x(h_{|t}, \sigma^*)$ allocates $B^H(h_{|t})$ δ^H-optimally across $[t, t^o)$ and $B^L(h_{|t})$ δ^L-optimally across $[t^o, T)$, where the "regime-change point" $t^o \in [t, T]$ uniquely renders collective spending continuous at t^o.

Equilibrium schedules given placement of second announcement

Given a type 4 node $h_{|t}$, let $\xi \equiv \hat{\xi}(h_{|t}) \in (t, T)$ denote the time of the subsequent node, and let $t^o \in [t, T]$ denote the regime-change point of x^o, the open-loop schedule of $\{B^i_t\}$ across $[t, T)$.

If $B^i(h_{|t}) = 0$ for either i, the unique equilibrium schedule across $[t, T)$ is trivial, with any player $-i$ with a positive budget allocating it δ^{-i}-optimally across this interval. We will therefore assume until the end of this subsection of the proof that $B^i(h_{|t}) > 0$ for both i: i.e. that $t^o \in (t, T)$.

We will now show that, for each possible value of ξ, there is an equilibrium schedule following $h_{|t}$. We will then show that the equilibrium schedule we have found is unique.

If $t^o \leq \xi$, x^o is an equilibrium schedule.

If L deviates from implementing x^o, his deviation has no impact on H's spending at any time, because $x^{H(o)}$ spends down B^H_t by ξ. By definition, $x^{L(o)}$ is L's unique best response to $x^{H(o)}$.

If H deviates in a way that maintains $B^H_\xi = 0$, her deviation likewise has no impact on L's spending at any time, so must lower her payoff, because $x^{H(o)}$ is her unique best response to $x^{L(o)}$. If she deviates such that $B^H_\xi > 0$, she must in equilibrium increase spending across $[\xi, T)$, because she induces the open-loop schedule following ξ with a higher initial value of B^H (positive instead of zero). Because the δ^H-discounted marginal utility to any allocation in this interval (from

X^o) is strictly lower than that to an allocation in $[t, t^o)$, and $u(\cdot)$ is convex, any deviation yielding $B^H_\xi > 0$ thus also lowers her payoff.

If $t^o > \xi$, let $\tilde{x}$ denote a schedule that maximizes U^H subject to the conditions that

a) H spends some budget $\leq B^H_t$ δ^H-optimally across $[t, \xi)$, and is the only spender across this interval and

b) the schedule across $[\xi, T)$ is the open-loop schedule across this interval given initial budgets $B^H_\xi, B^L_\xi = B^L_t e^{r(\xi - t)}$, i.e. $x^o[\xi, T, B^H_\xi, B^L_\xi]$.

To prove that such a schedule exists, let $\tilde{x}(B^H_\xi)$ denote the unique $\tilde{x}$ compatible with (a), (b), and a given value of B^H_ξ. Since U^H is continuous in B^H_ξ and the range of feasible values of B^H_ξ is $[0, B^H_t e^{r(\xi - t)}]$, which is compact, there is a U^H-maximizing $\tilde{x}$, by the extreme value theorem. Also, $\tilde{x}$ must satisfy

$$u'(\tilde{x}^H_t) \geq e^{(r - \delta^H)(\xi - t)} \frac{\partial U^H(x^o[\xi, T, B^H_\xi, B^L_\xi])}{\partial B^H_\xi}, \tag{108}$$

with equality if $B^H_\xi > 0$, or else H could increase her payoff from the $\tilde{x}$ baseline, while maintaining (a) and (b), by marginally shifting resources across ξ.

$\tilde{x}$ falls discontinuously at ξ. If it does not and $B^H_\xi = 0$, so that L begins spending immediately, this violates the stipulation that $t^o > \xi$. If it does and $B^H_\xi > 0$, then (108) cannot hold. Given $B^H_\xi > 0$, H must strictly prefer marginal *allocations* at ξ to marginal allocations just before ξ. This is because, if H reduces spending before ξ and thus increases B^H_ξ, she cannot allocate these resources to periods just after ξ without affecting $t^o[\xi, T, ...]$. Rather, by increasing B^H_ξ, she delays the regime-change point of the open-loop schedule beginning at ξ, increasing spending not only before t^o but also at times after t^o at which the δ^H-discounted marginal utility of resource allocation is lower.

$\tilde{x}$ is an equilibrium schedule. Any deviation at $h_{|t}$ by H that leaves B^H_ξ unchanged would simply consist of a strictly dispreferred allocation of her resources before ξ, and would have no effect on L's schedule. Any deviation by H that changes B^H_ξ, while maintaining δ^H-optimal spending before ξ (and of course the open-loop schedule after ξ), would leave (108) (with equality if $B^H_\xi > 0$) unsatisfied, and so would offer H an opportunity to increase her payoff by shifting resources across ξ in one direction or the other.

Any deviation by L to spending before ξ would decrease spending across $[\xi, T)$. Since the δ^L-discounted marginal utility of allocating resources to any time in this interval (even from x) is higher than that to any time in $[t, \xi)$, this deviation must lower L's payoff.

To see that there are no equilibrium schedules beyond those identified above, first observe that there cannot be t^L, t^H with

$$t \leq t^L < t^H, \quad x^i_{t^i}(h_{|t}, \sigma^*) > 0 \ \ \forall i. \tag{109}$$

To see this, suppose by contradiction that (109) obtains. Since $\alpha^H > \alpha^L$, we must have either

$$X_{t^H}(h_{|t}, \sigma^*) < e^{(r-\alpha^L)(t^H - t^L)} X_{t^L}(h_{|t}, \sigma^*) \tag{110}$$

$$\text{or} \quad X_{t^H}(h_{|t}, \sigma^*) > e^{(r-\alpha^H)(t^H - t^L)} X_{t^L}(h_{|t}, \sigma^*). \tag{111}$$

If $\xi \leq t^L$, (109) is incompatible with open-loop behavior after the second announcement.

If $\xi > t^H$, marginal reallocations from just after t^L to just after t^H, or vice-versa, do not affect budget sizes at ξ. They therefore do not affect the schedule implemented from ξ onward. But if (110), by the right-continuity of each player's spending, L prefers a marginal reallocation from just after t^L to just after t^H. Likewise, if (111), H prefers a marginal reallocation from just after t^H to just after t^L. Since one of these reallocations must increase a player's payoff across $[t, \xi)$ without affecting it from ξ onward, (109) is incompatible with equilibrium behavior at $h_{|t}$ given $\xi > t^H$.

If $\xi \in (t^L, t^H]$, let

$$\lambda(\epsilon) \equiv \sup_{s \in (\xi - \epsilon, \xi)} X_s, \quad \lambda \equiv \lim_{\epsilon \to 0} \lambda(\epsilon). \tag{112}$$

Because $\lambda(\epsilon)$ is weakly decreasing in ϵ, the limit is defined by the monotone convergence theorem.

If $\lambda < X_\xi$, then H can increase her payoff by increasing spending marginally for some period just before ξ. This marginally reduces B^H_ξ and so marginally reduces spending at all $s \in [\xi, T)$, and spending offers weakly lower δ^H-discounted marginal utility throughout this interval than at ξ.

If $\lambda \geq X_\xi$ and

$$\forall \epsilon > 0 \ \ \exists \, s \in (\xi - \epsilon, \xi) : x^L_s > 0, \tag{113}$$

then L can increase his payoff by reducing spending marginally for some period before ξ. This marginally increases B^L_ξ and so marginally increases spending at all $s \in [\xi, T)$, and allocation offers weakly higher δ^L-discounted marginal utility throughout this interval than at ξ.

If $\lambda \geq X_\xi$ and (113) fails, then either there exists $s \in [t, \xi)$ with $X_s = 0$ or spending across $[t, \xi)$ is not polarized: i.e. there exists $s \in (t^L, \xi)$ with $x^H_s > 0$. Both possibilities are incompatible with equilibrium behavior at $h_{|t}$ because the equilibrium schedule across $[t, \xi)$ must be the open-loop allocation of whatever resources are spent on this interval.

We have now shown that, in equilibrium, there do not exist t^L, t^H satisfying (109). That is, the time t^* at which H exhausts her budget is also the earliest time at which L spends (on $[t, T)$). We also know that the equilibrium schedule is an open-loop schedule before and after ξ. We will now see that these conditions rule out equilibrium schedules other than x^o given $t^o \leq \xi$ and $\tilde{x}$ (a U^H-maximizing schedule satisfying (a) and (b), not shown to be unique) given $t^o > \xi$.

Given $t^o \leq \xi$, a schedule $x \neq x^o$ satisfying the above conditions must feature $t^* < t^o$ or $t^* > t^o$.

If $t^* < t^o$, then $x_s^H > x_s^{H(o)}$ for $s < t^*$ (or else H does not exhaust her budget under x, and could increase her payoff by increasing spending at any time before ξ). Spending at t^* must be continuous, because if it is not, one of the players can increase their payoff by shift spending across t^* without affecting the other's spending at any time. x^L is then unaffordable for L, since it requires L to begin spending sooner and from a higher present-value spending rate—unless x^L falls discontinuously at ξ, which cannot happen in equilibrium, since L could then increase his payoff by smoothing his spending across $[t^*, T)$.

If $t^* \in (t^o, \xi)$, it can be shown precisely as above that either x^L fails to exhaust L's budget or x^L increases discontinuously at ξ, neither of which is compatible with equilibrium.

If $t^* \geq \xi$ and spending falls discontinuously at ξ, x^L fails to exhaust L's budget. If spending does not fall discontinuously at ξ, then, as shown in the construction of $\tilde{x}$, H can increase his payoff by increasing spending before ξ and lowering B_ξ^H.

Given $t^o > \xi$, if $t^* \geq \xi$ and x does not maximize H's payoff subject to (a) and (b), H can increase her payoff by setting $x^H = \tilde{x}^H$ across $[t, \xi)$ for some schedule $\tilde{x}$ that is U^H-maximizing given (a) and (b).

If $t^* < \xi$, the open-loop schedule is either unaffordable for L even up to ξ or x^L must fall discontinuously at ξ. In the latter case, L could profitably deviate by smoothing his spending across $[t^*, T)$ without affecting H's spending anywhere.

Equilibrium placement of second announcement

Let $h_{|t}^-$ be a type 3 node, and let $x^o \equiv x^o[t, T, B_t^H, B_t^L]$.

If $\xi_{\mathcal{T}}^L(h_{|t}^-) = \emptyset$ (and $\xi_{\mathcal{T}}^H(h_{|t}^-) = t$), it follows from the previous section that L implements x^o for any choice of announcement $\sigma^L(h_{|t}^-) \geq t^o$, and some instantiation of "$\tilde{x}$" for $\sigma^L(h_{|t}^-) < t^o$. Relative to x^o, therefore, any choice of $\sigma^L(h_{|t}^-) < t^o$ induces less spending across $[\sigma^L(h_{|t}^-), T)$ and greater spending across $[t, \sigma^L(h_{|t}^-))$, a reallocation which L disprefers. We must therefore have $\sigma^{L*}(h_{|t}) \geq t^o$, and $x(h_{|t}^-, \sigma^*) = x^o$.

If $\xi_{\mathcal{T}}^H(h_{|t}^-) = \emptyset$ (and $\xi_{\mathcal{T}}^L(h_{|t}^-) = t$), it likewise follows from the previous section that for any choice of $\sigma^H(h_{|t}^-)$, H implements an IC-polarized schedule.

Recall that H's favorite feasible IC-polarized schedule x^* is unique, with $t^* < t^o$. We therefore have $\sigma_s^{L*}(h_{|t}^+) = 0$ for $s \in [t, t^*)$, where $h_{|t}^+$ denotes the type 4 node subsequent to $h_{|t}^-$ given $\sigma^H(h_{|t}^-) = t^*$. So H can indeed achieve x^* with this choice of announcement, and cannot achieve a superior schedule.

This identifies the unique equilibrium schedule following $h_{|t}^-$.

Unique equilibrium schedule given placement of first announcement

Let $h_{|\overline{\tau}}$ be a type 2 node and $x^o \equiv x^o[\overline{\tau}, T, B_{\overline{\tau}}^H, B_{\overline{\tau}}^L]$. Note that t^o is interior because we have assumed $B_{\overline{\tau}}^i > 0$ for both i. Let $\xi \equiv \hat{\xi}(h_{|\overline{\tau}})$.

If $\xi_{\overline{\tau}}^H(h_{|\overline{\tau}}) = \xi$ is defined, the unique equilibrium schedule across $[\xi, T)$ is $x^o[\xi, T, B_\xi^H, B_\xi^L]$. So, by a proof identical to that characterizing equilibrium schedules following a type 4 node, any equilibrium schedule following $h_{|\overline{\tau}}$ is IC-polarized.

If $\xi_{\overline{\tau}}^H(h_{|\overline{\tau}}) = \emptyset$, so that $\xi_{\overline{\tau}}^L(h_{|\overline{\tau}}) = \xi$, the unique equilibrium schedule across $[\xi, T)$ is $x^*[\xi, T, B_\xi^H, B_\xi^L]$. Furthermore, because x^* depends only on the initial budgets and not on other features of the history, the unique equilibrium schedule across $[t, \xi)$ is the open-loop schedule for whatever pair of budgets is allocated to this interval. Given this pair of observations, we will now show that any equilibrium schedule following $h_{|\overline{\tau}}$ is IC-polarized.

Suppose $x(h_{|\overline{\tau}}, \sigma^*)$ is not IC-polarized. It must then consist of an open-loop schedule before ξ, on which L spends, followed by a Stackelberg schedule on and after ξ, on which H spends.

If $x(h_{|\overline{\tau}}, \sigma^*)$ is not IC-polarized and $\lambda \geq X_\xi(h_{|\overline{\tau}}, \sigma^*)$, where λ is defined as in (112), we will show that L can increase his payoff by reducing spending before ξ. We will show this in two steps.

First, we will show that the δ^L-discounted utility generated by a resource allocation across the interval before ξ during which L spends is strictly less than the δ^L-discounted utility lost by an equal decrease to $e^{-r(\xi - \overline{\tau})} B_\xi^H$. That is, L would prefer to invest marginal resources until ξ and then to transfer them to H (if this were permitted).

Second, we will show that if L invests to ξ, given that $\sigma^H = \sigma^{H*}$ from ξ onward, he can implement the collective schedule that would result by transferring the resources to H at ξ and then following strategy σ^{L*}. This will imply that the δ^L-discounted utility lost by decreasing H's budget at ξ is weakly less than that lost by equally decreasing L's budget at ξ. In conjunction, these results imply that L strictly prefers to reduce spending before ξ, as desired.

For the first step, given a time $s < \xi$ at which L spends, the δ^L-discounted flow utility to a marginal resource allocation at s equals

$$m^L \equiv e^{(r-\delta^L)(s-t)}(\sigma_s^{L*}(h_{|\overline{\tau}}))^{-\gamma}$$

per unit allocated.

As noted in the discussion of Stackelberg schedules at the beginning of this proof, a decrease to B^H_ξ decreases $t^*[\xi, T, B^H_\xi, B^L_\xi]$.

A decrease to B^H_ξ does not however decrease t^* by enough that

$$x^H_\xi = \frac{B^H_\xi \alpha^H}{1 - e^{-\alpha^H(t^*-\xi)}} \tag{114}$$

weakly rises. To see this, choose two potential values for H's budget at ξ, $\underline{B}^H < \overline{B}^H$, and suppose $t^*[\underline{B}^H] < t^*[\overline{B}^H]$ are such that

$$\frac{\underline{B}^H \alpha^H}{1 - e^{-\alpha^H(t^*[\underline{B}^H]-\xi)}} > \frac{\overline{B}^H \alpha^H}{1 - e^{-\alpha^H(t^*[\overline{B}^H]-\xi)}}. \tag{115}$$

For $Z = A, B, C, D$, let $\overline{Ⓩ}$ denote the value of Ⓩ at $B^H = \overline{B}^H$, $\tilde{t} = t^*[\overline{B}^H]$, and $t = \xi$ (as ξ is now the beginning of the interval), fixing B^L. Define $\underline{Ⓩ}$ likewise, but at $B^H = \underline{B}^H$ and $\tilde{t} = t^*[\underline{B}^H]$. If the two sides of (115) were equal, a proof precisely analogous to that from (103) to just following (107) would yield

$$\underline{Ⓑ}(\overline{Ⓒ} - \underline{Ⓒ}) + (\overline{Ⓓ} - \overline{Ⓒ})(\underline{Ⓑ} - \overline{Ⓑ}) + (1 - \underline{Ⓑ})(\overline{Ⓓ} - \underline{Ⓓ}) < 0, \tag{116}$$

with $\overline{Ⓒ} - \underline{Ⓒ}$, $\underline{Ⓑ} - \overline{Ⓑ}$, and $\overline{Ⓓ} - \underline{Ⓓ}$ negative, and the other terms positive. Inequality (115) then increases the absolute values of these three differences without affecting the positive terms. So (116) holds.

Thus, when H begins at ξ with budget $\underline{B}^H < \overline{B}^H$, if she exhausts her budget at a time so far below $t^*[\overline{B}^H]$ that (115) holds, then the marginal utility for H to increasing the regime-change point from $t^*[\underline{B}^H]$ is greater than the marginal utility for H to increasing the regime-change point from $t^*[\overline{B}^H]$ given initial budget $\overline{B}^H$. But both must be zero. This concludes the proof that when B^H_ξ is smaller, H's spending path from ξ to the chosen regime-change point $t^*[B^H_\xi]$ is lower.

H's spending path before t^* falls, the point t^* at which spending falls discontinuously itself decreases, and L's spending path after t^* falls. It follows that, as in the open-loop case, a decrease to B^H_ξ decreases the resources allocated to all periods from ξ to T in equilibrium. Furthermore, the δ^L-discounted flow utility to a marginal resource allocation at $s \geq \xi$ increases in s (at proportional rate $\delta^H - \delta^L$) from ξ to t^*, then jumps and is constant from t^* to T. Since, flow utility is strictly concave in flow spending, the δ^L-discounted utility (as of t) lost by a decrease to $e^{-r(\xi-t)}B^H_\xi$ is strictly more than m^L per unit decreased. This proves that if at t L allocates any resources before ξ, and collective spending weakly falls at ξ, then L would prefer on the margin to invest resources until ξ and then to transfer them to H (if this were permitted).

We have just seen that, when B^H_ξ increases, x^H_ξ increases. For the second step, we will first show that, when B^L_ξ increases, x^H_ξ increases by weakly less than when B^H_ξ increases.

Suppose that it does not. Then, given a partial schedule $x_{|\xi}$, for simplicity let B^i denote $B^i(x_{|\xi})$,

$$U^H(B^H, B^L) \equiv \int_\xi^T e^{-\delta^H(s-\xi)} u(X_t(x_{|\xi}, \sigma^*))ds, \tag{117}$$

and

$$V^H(B^H, B^L, t^*)$$

denote H's payoff following ξ given B^H, B^L, and (not necessarily optimal) regime-change time t^*, as defined by (101) (with ξ, t^* replacing $t, \tilde{t}$). By the envelope theorem, when at ξ H optimally sets the regime-change point at $t^*[B^H, B^L]$, since her payoff after ξ is $\mathcal{C}^1$ in both B^H and the regime-change point, the marginal utility to H of having a larger budget is always the same as what it would be if the regime-change point were fixed:

$$\frac{\partial U^H}{\partial B^H}(B^H, B^L) = \frac{\partial V^H}{\partial B^H}\big(B^H, B^L, t^*[B^H, B^L]\big) = (x^H_\xi)^{-\gamma}. \tag{118}$$

So, on the assumption that when B^L_ξ increases, x^H_ξ increases by more than when B^H_ξ increases, we have

$$\frac{\partial^2 U^H}{\partial B^{H2}} > \frac{\partial^2 U^H}{\partial B^L \partial B^H}. \tag{119}$$

It can be seen from (100) that if both B^i are multiplied by the same proportion (say m), then $\partial U^H/\partial t^*$ is multiplied by the same proportion for all t^* ($m^{1-\gamma}$). The unique t^* setting $\partial U^H/\partial t^* = 0$ therefore does not change, and spending is multiplied by m at all periods. That is, $t^*[B^H, B^L]$ is h.o.d. 0 and $x^*[B^H, B^L]$ is h.o.d. 1. It follows from this homotheticity that $\partial U^H/\partial B^i$ is in turn multiplied by the same proportion ($m^{-\gamma}$) for both B^i. Formally,

$$\frac{\frac{\partial^2 U^H}{\partial B^L \partial B^H} B^L + \frac{\partial^2 U^H}{\partial B^{H2}} B^H}{\frac{\partial U^H}{\partial B^H}} = \frac{\frac{\partial^2 U^H}{\partial B^{L2}} B^L + \frac{\partial^2 U^H}{\partial B^H \partial B^L} B^H}{\frac{\partial U^H}{\partial B^L}}. \tag{120}$$

By Young's Theorem,

$$\frac{\partial^2 U^H}{\partial B^L \partial B^H} = \frac{\partial^2 U^H}{\partial B^H \partial B^L}.$$

Substituting the left-hand side for the right-hand side into (120) and rearranging,

$$\frac{\partial^2 U^H}{\partial B^L \partial B^H}\Big(B^L \frac{\partial U^H/\partial B^L}{\partial U^H/\partial B^H} - B^H\Big) + \frac{\partial^2 U^H}{\partial B^{H2}} B^H \frac{\partial U^H/\partial B^L}{\partial U^H/\partial B^H} = \frac{\partial^2 U^H}{\partial B^{L2}} B^L.$$

Then given (119),

$$\frac{\partial^2 U^H}{\partial B^L \partial B^H} \frac{\partial U^H/\partial B^L}{\partial U^H/\partial B^H} + \frac{B^H}{B^L} \frac{\partial^2 U^H}{\partial B^L \partial B^H}\Big(\frac{\partial U^H/\partial B^L}{\partial U^H/\partial B^H} - 1\Big) < \frac{\partial^2 U^H}{\partial B^{L2}}. \tag{121}$$

Given B^H_ξ, B^L_ξ, if resources are transferred at ξ from L to H, H can always implement the collective schedule that would have obtained without the transfer, by adopting her original schedule until the original regime-change point $t^*[B^H_\xi, B^L_\xi]$ and then following L's original schedule after $t^*[B^H_\xi, B^L_\xi]$ until her funds are exhausted. H can then strictly improve on this schedule by smoothing her spending around $t^*[B^H_\xi, B^L_\xi]$. This implies that

$$\frac{\partial U^H}{\partial B^H} > \frac{\partial U^H}{\partial B^L}. \tag{122}$$

The term of (121) in parentheses is therefore negative. Since flow spending by H at ξ increases as B^L increases, the envelope theorem gives us that

$$\frac{\partial^2 U^H}{\partial B^L \partial B^H} < 0.$$

The second term of the sum in (121) is therefore positive.

In combination with (122), which implies that the coefficient on $\partial^2 U^H/(\partial B^L \partial B^H)$ in the first term of the sum is less than 1, this gives us

$$\frac{\partial^2 U^H}{\partial B^L \partial B^H} < \frac{\partial^2 U^H}{\partial B^{L2}}. \tag{123}$$

By (119) and (123), if marginal resources are transferred from H to L at ξ (from some initial values of B^H and B^L), $\partial U^H/\partial B^H$ rises by

$$\frac{\partial^2 U^H}{\partial B^L \partial B^H} - \frac{\partial^2 U^H}{\partial B^{H2}} < 0 \tag{124}$$

(i.e. falls) per unit transferred, and $\partial U^H/\partial B^L$ rises by

$$\frac{\partial^2 U^H}{\partial B^{L2}} - \frac{\partial^2 U^H}{\partial B^L \partial B^H} > 0. \tag{125}$$

As seen above with respect to B^H, by the envelope theorem,

$$\frac{\partial U^H}{\partial B^L}(B^H, B^L) = \frac{\partial V^H}{\partial B^L}\big(B^H, B^L, t^*[B^H, B^L]\big).$$

Partially differentiating (101) with respect to B^L, with $\xi, t^*[B^H, B^L]$ replacing $t, \tilde{t}$ respectively, we have

$$\big(B^L\big)^{-\gamma}\frac{1 - e^{(\delta^L-\alpha^L-\delta^H)(T-t^*[B^H,B^L])}}{\delta^H + \alpha^L - \delta^L} e^{-\gamma\alpha^H(t^*[B^H,B^L]-\xi)}\left(\frac{\alpha^L}{1 - e^{-\alpha^L(T-t^*[B^H,B^L])}}\right)^{1-\gamma}.$$

Given (125), there exists an $\epsilon > 0$ such that, when B^H falls by ϵ and B^L increases by ϵ, this partial derivative rises:

$$\begin{aligned}&\big(B^L+\epsilon\big)^{-\gamma}\frac{1 - e^{(\delta^L-\alpha^L-\delta^H)(T-t^*[B^H-\epsilon,B^L+\epsilon])}}{\delta^H + \alpha^L - \delta^L}\cdot \\ &e^{-\gamma\alpha^H(t^*[B^H+\epsilon,B^L-\epsilon]-\xi)}\left(\frac{\alpha^L}{1 - e^{-\alpha^L(T-t^*[B^H+\epsilon,B^L-\epsilon])}}\right)^{1-\gamma} \qquad (126)\\ \geq\ &\big(B^L\big)^{-\gamma}\frac{1 - e^{(\delta^L-\alpha^L-\delta^H)(T-t^*[B^H,B^L])}}{\delta^H + \alpha^L - \delta^L} e^{-\gamma\alpha^H(t^*[B^H,B^L]-\xi)}\left(\frac{\alpha^L}{1 - e^{-\alpha^L(T-t^*[B^H,B^L])}}\right)^{1-\gamma}\end{aligned}$$

$$\iff \frac{(B^L+\epsilon)\alpha^L}{1 - e^{-\alpha^L(T-t^*[B^H-\epsilon,B^L+\epsilon])}} \leq \frac{B^L\alpha^L}{1 - e^{-\alpha^L(T-t^*[B^H,B^L])}} e^{\alpha^H(t^*[B^H,B^L]-t^*[B^H-\epsilon,B^L+\epsilon])} \times \left(\frac{\Big(1 - e^{(\delta^L-\alpha^L-\delta^H)(T-t^*[B^H,B^L])}\Big)\Big/\Big(1 - e^{-\alpha^L(T-t^*[B^H,B^L])}\Big)}{\Big(1 - e^{(\delta^L-\alpha^L-\delta^H)(T-t^*[B^H-\epsilon,B^L+\epsilon])}\Big)\Big/\Big(1 - e^{-\alpha^L(T-t^*[B^H-\epsilon,B^L+\epsilon])}\Big)}\right)^{-\frac{1}{\gamma}}.$$

Because we have seen that both decreases to B^H and increases to B^L lower t^*,

$$t^*[B^H-\epsilon, B^L+\epsilon] < t^*[B^H, B^L]. \qquad (127)$$

It follows that the fraction in the numerator of the large fraction just above exceeds the fraction in the denominator, and thus that

$$\frac{(B^L+\epsilon)\alpha^L}{1 - e^{-\alpha^L(T-t^*[B^H-\epsilon,B^L+\epsilon])}} < e^{\alpha^H(t^*[B^H,B^L]-t^*[B^H-\epsilon,B^L+\epsilon])}\frac{B^L\alpha^L}{1 - e^{-\alpha^L(T-t^*[B^H,B^L])}}. \qquad (128)$$

Let Ⓐ$_{\text{pre}}$ (for "pre-transfer") denote the value of Ⓐ at B^H, B^L, and $t^*[B^H, B^L]$; Ⓐ$_{\text{post}}$ denote the value of Ⓐ at $B^H-\epsilon$, $B^L+\epsilon$, and $t^*[B^H-\epsilon, B^L+\epsilon]$; and Ⓑ$_{\text{pre/post}}$–Ⓓ$_{\text{pre/post}}$ likewise. We will now show that, given (124) and (128),

$$\begin{aligned}&Ⓑ_{\text{pre}}\big(Ⓒ_{\text{post}} - Ⓒ_{\text{pre}}\big) + \big(Ⓓ_{\text{post}} - Ⓒ_{\text{post}}\big)\big(Ⓑ_{\text{pre}} - Ⓑ_{\text{post}}\big)\\ &+ \big(1 - Ⓑ_{\text{pre}}\big)\big(Ⓓ_{\text{post}} - Ⓓ_{\text{pre}}\big) > 0. \qquad (129)\end{aligned}$$

We will do this by showing that each of the terms of the sum above is positive.

We know from the discussion surrounding (104) that Ⓑ$_{\text{pre}}$ and $1 -$ Ⓑ$_{\text{pre}}$ are positive. Also, (126)–(127) and (124) give us that Ⓒ$_{\text{post}} >$ Ⓒ$_{\text{pre}}$.

It follows from the paragraph ending at (106) that Ⓓ$_{\text{pre}} >$ Ⓒ$_{\text{pre}}$. It follows from (128) and (124) that Ⓑ$_{\text{pre}} >$ Ⓑ$_{\text{post}}$. It follows in turn from this and the reasoning surrounding (107) that Ⓓ$_{\text{post}} >$ Ⓓ$_{\text{pre}}$.

Recall that, by definition, the marginal utility for H of raising t^* from $t^*[B^H, B^L]$ (given that the players' budgets equal B^H, B^L) equals zero. It thus follows from (129), by the calculations preceding (103) and (105), that the marginal utility for H of raising t^* from $t^*(B^H - \epsilon, B^L + \epsilon)$ (given that the players' budgets equal $B^H - \epsilon$ and $B^L + \epsilon$ respectively) is positive. This contradicts the supposition that setting $t^*(B^H - \epsilon, B^L + \epsilon)$ is optimal for H given these budgets.

This completes the proof that (119) is false, i.e. that in fact

$$\frac{\partial^2 U^H}{\partial B^{H2}} \leq \frac{\partial^2 U^H}{\partial B^L \partial B^H},$$

and thus that, when B^L_ξ increases, x^H_ξ increases by weakly less than when B^H_ξ increases.

So, if L reduces spending before ξ, thereby increasing B^L_ξ, he can implement the schedule that would have subsequently have been implemented if the additional resources had been transferred to H at ξ. In combination with the result that L would prefer to reduce spending before ξ if the resources saved would then be transferred to H at ξ, this completes the proof that an equilibrium schedule following $h_{|\overline{\tau}}$ cannot both fail to be IC-polarized and feature a weak decrease at $\xi \equiv \hat{\xi}(h_{|\overline{\tau}})$.

The proof that the schedule implemented by σ^* following $h_{|\overline{\tau}}$ cannot both fail to be IC-polarized and feature $\lambda < X_\xi(h_{|\overline{\tau}}, \sigma^*)$ is more straightforward.

If an equilibrium σ^* implements a non-IC-polarized schedule following $h_{|\overline{\tau}}$, we must have

$$\begin{aligned}\big(X_{\overline{\tau}}(h_{|\overline{\tau}}, \sigma^*)\big)^{-\gamma} &\leq e^{(r-\delta^H)(\xi-\overline{\tau})} \frac{\partial U^H(B^H, B^L)}{\partial B^H} \\ &= e^{(r-\delta^H)(\xi-\overline{\tau})} \big(X_\xi(h_{|\overline{\tau}}, \sigma^*)\big)^{-\gamma},\end{aligned} \tag{130}$$

where, as before, $B^i \equiv B^i_\xi(h_{|\overline{\tau}}, \sigma^*)$ and $U^H(\cdot)$ is defined as in (117). The weak inequality must hold for H not to prefer to increase spending at $\overline{\tau}$ at the cost of lowering her budget at ξ. The equality of the second line follows from the envelope theorem.

But the δ^H-discounted marginal utility of a resource allocation at s, $e^{(r-\delta^H)(s-\overline{\tau})}\big(X_s\big)^{-\gamma}$, weakly falls across $[\overline{\tau}, \xi)$ and, given a jump in collective spending at ξ, falls discontinuously at ξ. So (130) cannot hold.

This completes the proof that any equilibrium schedule following $h_{|\overline{\tau}}$ is IC-polarized.

Unique equilibrium schedule from beginning of period

Let $h^-_{|\overline{\tau}}$ be a type 1 node. Suppose H chooses announcement $\sigma^H(h^-_{|\overline{\tau}}) = t^* \equiv t^*[\overline{\tau}, T, B^H_{\overline{\tau}}, B^L_{\overline{\tau}}]$. We will now show that this subsequently implements the Stackelberg schedule $x^* \equiv x^*[\overline{\tau}, T, B^H_{\overline{\tau}}, B^L_{\overline{\tau}}]$, regardless of $\sigma^L(h^-_{|\overline{\tau}})$ (given that the players will play an equilibrium strategy profile after $h^-_{|\overline{\tau}}$). Since this is H's favorite feasible IC-polarized schedule, this will establish that it is the unique equilibrium schedule following $h^-_{|\overline{\tau}}$. We will let $h^+_{\overline{\tau}}$ denote the node after $h^-_{|\overline{\tau}}$ given the players' announcements.

Regardless of $\sigma^L(h^-_{|\overline{\tau}})$, $\hat{\xi} \equiv \hat{\xi}(h^-_{|\overline{\tau}}) \leq t^*$. Then if σ is an equilibrium, the schedule following $h^+_{|\overline{\tau}}$ must be

- open-loop across $[\overline{\tau}, \hat{\xi})$ (for some non-negative pair of budgets allocated by the players to this interval) and
- IC-polarized.

At $h^+_{|\overline{\tau}}$, therefore, L cannot plan to spend in equilibrium. Because $\hat{\xi} \leq t^* < t^o$, H's optimal open-loop spending plan across $[\overline{\tau}, \hat{\xi})$ that is continuous with L's δ^L-optimal spending up to T would not exhaust H's budget.

Since L spends nothing before $\hat{\xi}$, H can implement $x^*[\overline{\tau}, T, B^H_{\overline{\tau}}, B^L_{\overline{\tau}}]$ by following this schedule up to $\hat{\xi}$ and, at $\hat{\xi}$, if $\hat{\xi} < t^*$, announcing t^* again. This is H's favorite feasible IC-polarized schedule, so this plan must be H's best response to $\sigma^L(h^-_{|\overline{\tau}})$, and what obtains in equilibrium.

Inductive step

Given a grid point $\tau < \overline{\tau}$, suppose that the unique equilibrium schedule across $[\tau', T)$ is Stackelberg. We will show that the unique equilibrium schedule beginning at any pre-announcement node $h_{|\tau}$ is Stackelberg as well.

As before, given a node $h_{|t}$, if $B^i(h_{|t}) = 0$ for some i, the unique subgame equilibrium following $h_{|t}$ is trivial: $-i$ allocates $B^i(h_{|t})$ δ^{-i}-optimally across $[t, T)$. We therefore assume for the rest of this section of the proof that, when studying the game beginning at any node $h_{|t}$, $B^i(h_{|t}) > 0$ for both i.

Unique equilibrium schedule after second announced node

Let $h_{|t}$ be a type 5 node following the second announcement after τ. Let $x^o \equiv x^o[t, T, B^H_t, B^L_t]$ and $x^* \equiv x^*[t, T, B^H_t, B^L_t]$.

A schedule implemented by σ^* following $h_{|t}$ must be IC-polarized, by the proof above of the unique equilibrium schedule following a type 2 node in the last period given $\xi^L > \xi^H$. It must also be Stackelberg across $[\tau', T)$, and, since the Stackelberg schedule following τ' depends only on the budget sizes at τ', it must be open-loop across $[t, \tau')$.

Let $\tilde{t}$ denote the time at which H exhausts her budget on a schedule implemented by σ^*.

If $t^o \leq \tau'$, the unique equilibrium schedule following $h_{|t}$ is x^o.

The fact that no other schedule is an equilibrium schedule follows from polarization. If $\tilde{t} > t^o$, then collective spending increases discontinuously at $\tilde{t}$. Whenever this occurs, L can then increase his payoff by beginning to spend before $\tilde{t}$. If $\tilde{t} < t^o \leq \tau'$, then collective spending must fall discontinuously at $\tilde{t}$, so H can increase her payoff by delaying some spending to after $\tilde{t}$. Neither deviation affects the other player's spending.

Any deviation by L reallocates resources from times with higher to times with weakly lower δ^L-discounted marginal utility to resource allocation without affecting H's spending at any time. A deviation by H that maintains $B^H_{\tau'} = 0$ reallocates resources from times with higher to times with weakly lower δ^H-discounted marginal utility to resource allocation without affecting L's spending at any time. A deviation by H that results in $B^H_{\tau'} > 0$ would increase spending across $[\tau', T)$. Since the δ^H-discounted marginal utility to resource allocation throughout this interval is strictly lower than that to resource allocation before τ', such a deviation must lower H's payoff. x^o is therefore an equilibrium schedule.

If $t^* \geq \tau'$, the unique equilibrium schedule following $h_{|t}$ is x^*.

The fact that no other schedule is an equilibrium schedule follows from polarization. If $\tilde{t} < \tau'$, then collective spending falls discontinuously at $\tilde{t}$, so H can increase her payoff by delaying some spending to after $\tilde{t}$ without affecting L's spending. Given that $\tilde{t} \geq \tau'$, H can implement x^* by and only by setting $x^H_s = x^{H*}_s$ for $s \in [t, \tau')$. Since x^* is H's favorite feasible IC-polarized schedule, H must do this in equilibrium.

Any deviation by H would result in an IC-polarized schedule other than x^*, so H disprefers it. Any deviation by L must take the form of spending before τ'. As shown in the proof above of the unique equilibrium schedule following a type 2 node in the last period given $\xi^L > \xi^H$, L disprefers such a deviation. x^* is therefore an equilibrium schedule.

If $\tau' \in (t^*, t^o)$, the unique equilibrium schedule following $h_{|t}$ consists of H allocating B^H_t δ^H-optimally across $[t, \tau')$ and L allocating B^L_t δ^L-optimally across $[\tau', T)$. Call this the quasi-Stackelberg schedule with regime-change time τ'.

Again, the fact that no other schedule is an equilibrium schedule follows from polarization. If $\tilde{t} < \tau'$, then collective spending falls discontinuously at $\tilde{t}$, so H can increase her payoff by delaying some spending to after $\tilde{t}$ without affecting L's spending. H's favorite schedule among those with $\tilde{t} > \tau'$—given a Stackelberg schedule after τ', and thus a discontinuous drop in collective spending at $\tilde{t}$ and B_t^L spent δ^L-optimally across $[\tilde{t}, T)$—is that in which she spends B_t^H δ^H-optimally across $[t, \tilde{t})$. H's favorite location for $\tilde{t}$, under the restriction that $\tilde{t} \geq \tau'$, then follows from the proof that H has a unique favorite $\tilde{t}$ without this restriction, in the discussion of Stackelberg schedules at the beginning of this proof. There it is shown that, given that the derivative of H's payoff with respect to regime-change time $\tilde{t}$ (there denoted ξ) equals 0 at $\tilde{t} = t^*$, this derivative is negative for all $\tilde{t} > t^*$. Here, since $\tau' > t^*$, it follows that H's favorite location for $\tilde{t}$, under the restriction that $\tilde{t} \geq \tau'$, is τ'. So, from a strategy profile implementing an IC-polarized schedule in which H does not spend down by τ', H can implement a strictly preferred schedule by spending down δ^H-optimally by τ'.

This implies that H cannot profitably deviate. Since collective spending weakly falls at τ', it is clear that L also cannot profitably deviate. The quasi-Stackelberg schedule with regime-change time τ' is therefore an equilibrium schedule.

Equilibrium schedules given placement of second announcement

Given a type 4 node $h_{|t}$, let $\xi \equiv \hat{\xi}(h_{|t})$ denote the time of the subsequent node. Let $x^o \equiv x^o[t, T, B_t^H, B_t^L]$ and $x^* \equiv x^*[t, T, B_t^H, B_t^L]$. Recall that $t^* < t^o$.

To begin, recall that the unique equilibrium schedule after $h_{|\xi}$ is IC-polarized. Also, since it is determined by $\{B_\xi^i\}$, any equilibrium schedule following $h_{|t}$ is open-loop until ξ.

If $t^o < \xi$, the unique equilibrium schedule following $h_{|t}$ is x^o.

We will first show that no other schedule is an equilibrium schedule.

If L does not spend before ξ, collective spending jumps up at ξ. If H exhausts her budget at ξ, then L can increase his payoff by beginning to spend before ξ. If H spends at ξ, then spending jumps up at ξ, regardless of whether the following schedule is open-loop, Stackelberg, or quasi-Stackelberg. In all three cases, H can increase her payoff by marginally increasing spending to before ξ. In the open-loop case this is because this deviation will shift resources away from all times $[\xi, T)$ to times before ξ, and all the latter offer higher δ^H-discounted marginal utility than all the former; in the Stackelberg case, it follows from the envelope theorem; and in the quasi-Stackelberg case H can reallocate resources from after to before ξ without affecting L's schedule.

Suppose therefore that L spends before ξ. If H spends at ξ, then if collective spending weakly jumps up at ξ, H prefers to increase spending across some interval before ξ during which L spends, by the reasoning just above. If collective spending jumps down at ξ, L wants to decrease his spending marginally before ξ: in the open-loop case because this will constitute a transfer away from all of $[\xi, T)$, and so from lower- to higher- δ^L-discounted marginal utility times; in the Stackelberg case by the proof in the "Final period: Unique equilibrium placement of first announcement" section above; and in the quasi-Stackelberg case because he can then increase reallocate spending to after τ without affecting H's schedule. If instead H exhausts her budget before ξ, then spending must be continuous at ξ, or else L can increase his payoff by smoothing his spending without affecting H's spending. So only x^o can be an equilibrium schedule following $h_{|t}$.

To verify that x^o is in fact an equilibrium schedule, observe that any deviation by H that leaves $B_\xi^H > 0$ transfers resources from before ξ to all times weakly after ξ, which all offer strictly lower δ^H-discounted marginal utility. Any deviation thus lowers her payoff. It is trivial that L has no incentive to deviate.

If $t^* \leq \xi$ and $t^o \geq \xi$, the unique equilibrium schedule following $h_{|t}$ is the quasi-Stackelberg schedule with regime-change time ξ (reducing to the open-loop schedule if $t^o = \xi$ and to the Stackelberg schedule if $t^* = \xi$).

Suppose L spends before ξ. If collective spending jumps down at ξ, L prefers to reallocate marginal spending from before to after ξ regardless of which equilibrium-type prevails from ξ onward, as explained above. If collective spending does not jump down at ξ, then H must spend at ξ (since $t^o > \xi$), and H prefers to reallocate marginal spending from after to before ξ regardless of which equilibrium-type prevails from ξ onward, as explained above.

So in any equilibrium, L does not spend before ξ. H does not spend at or after ξ, because if she does, the equilibrium schedule must be IC-polarized with regime-change time $\tilde{t} > \xi$; H strictly disprefers any such schedule to the quasi-Stackelberg schedule with regime-change time ξ, as explained above; and H can implement the latter by exhausting her budget by ξ.

This is an equilibrium schedule because any deviation by H implements an IC-polarized schedule which she disprefers, and because again L has no incentive to deviate.

If $t^* \in (\xi, \tau')$ and $t^o \leq \tau'$, the unique equilibrium schedule following $h_{|t}$ is H's favorite IC-polarized schedule that is open-loop after ξ: i.e. the schedule denoted $\tilde{x}$ in the proof of the unique equilibrium schedule given the placement of the second announcement in the final period (see the paragraph up to (108)).

Suppose L spends before ξ. Then, as we have seen, because $t^o > \xi$, either spending jumps down at ξ or it does not and H spends at ξ; and in the former case L prefers to reallocate from before to after ξ, and in the latter case H prefers to reallocate in the other direction. So L does not spend before ξ on any equilibrium schedule.

Because $t^o \leq \tau'$, $t^o[\xi, T, B_\xi^H, B_\xi^L] \leq \tau'$ as long as H spends at least as quickly on

$[t, \xi)$ as she does on $x^o[t, T, B_t^H, B_t^L]$; so in this case the unique equilibrium schedule following ξ is open-loop. If H spends more slowly on $[t, \xi)$ than $x^{H(o)}[t, T, B_t^H, B_t^L]$, then $B_\xi^H > 0$ and collective spending jumps up at ξ (whichever of the three equilibria obtains), and H prefers to reallocate from after to before ξ as we have seen. So any equilibrium schedule must be IC-polarized and open-loop after ξ.

H's favorite schedule of this form is found (and found to exist and be unique) in the proof of the unique equilibrium schedule given the placement of the second announcement in the final period. Since it never features a jump up in collective spending at ξ, its regime-change time is before τ'. Given that L spends nothing before ξ, H can unilaterally implement it (as of t) simply by following it with her own spending up to ξ. So no other schedule can obtain in equilibrium.

Any deviation by L to spending before ξ reallocates resources from all times weakly after ξ to some times before. Since the former all offer strictly higher δ^L-discounted marginal utility to resource allocations than the latter, any deviation lowers L's payoff. Any deviation by H that leaves $t^o[\xi, T, B_\xi^H, B_\xi^L] \leq \tau'$ would by construction lower her payoff. Any more extreme deviation would implement a schedule that can be constructed by beginning with a schedule that leaves $t^o[\xi, T, B_\xi^H, B_\xi^L] \leq \tau'$ and reallocating resources from before ξ to some set of times weakly after ξ, all of which (from this baseline) offer strictly lower δ^H-discounted marginal utility. So H has no incentive to deviate either, and the proposed schedule is in fact an equilibrium schedule.

If $t^* \in (\xi, \tau')$ and $t^o > \tau'$, either only the quasi-Stackelberg schedule with regime-change time τ' is the unique equilibrium schedule following $h_{|t}$, only H's favorite IC-polarized schedule that is open-loop following ξ, or both are equilibrium schedules following $h_{|t}$.

L cannot spend before ξ in equilibrium just as in the $t^* \in (\xi, \tau')$, $t^o \leq \tau'$ case. Again, therefore, B_ξ^H determines the equilibrium schedule following ξ. If B_ξ^H is small (including zero), the equilibrium schedule following ξ is open-loop; if larger, it is quasi-Stackelberg with regime-change time τ'; and if larger still, it is Stackelberg, with a regime-change time after τ'.

If H spends so quickly before ξ that the equilibrium schedule following ξ is Stackelberg with a regime-change time after τ', then her (and collective) spending must jump up at ξ, since $t^* < \tau'$. H then prefers to reallocate from after to before ξ. This leaves the quasi-Stackelberg and open-loop possibilities. H can implement either, as we have seen. In equilibrium, she must implement her favorite of the two, or simply one of the two if they offer her equal payoffs.

Given that she does so, a deviation by L to spending before ξ never increases spending at any time after ξ, so it always shifts resources from times with higher to times with lower δ^L-discounted marginal utility, lowering his payoff. A deviation by H before ξ always lowers her payoff by construction. The proposed equilibrium schedules are therefore in fact equilibrium schedules.

If $t^* \geq \tau'$, x^* is the unique equilibrium schedule following $h_{|t}$.

The proof that L cannot spend before ξ in equilibrium is precisely as in the cases above. Since only H spends before ξ, and any equilibrium schedule after ξ is IC-polarized, any equilibrium schedule is IC-polarized. x^* is H's favorite feasible IC-polarized schedule, and she can implement it by following it up to ξ. So if there is an equilibrium schedule, it must be x^*.

Any deviation by H would implement an alternative, and therefore dispreferred, IC-polarized schedule. Any deviation by L would implement a reallocation from some or all times weakly after ξ to before ξ, which he disprefers. x^* is therefore an equilibrium schedule.

Unique equilibrium schedule incorporating choice of second announcement

Given a type 3 node $h_{|t}$, let $x^o \equiv x^o[t, T, B_t^H, B_t^L]$ and $x^* \equiv x^*[t, T, B_t^H, B_t^L]$.

Suppose $\xi_\tau^L(h_{|t}) = \emptyset$ (and $\xi_\tau^H(h_{|t}) = t$).

If $t^o \leq \tau'$, L implements x^o by choosing announcement $\xi^L \in [t^o, \tau']$; the quasi-Stackelberg schedule with regime-change time ξ by choosing $\xi^L \in (t^*, t^o)$; and H's favorite schedule that is open-loop after ξ by choosing $\xi^L \in (t, t^*]$ (which is Stackelberg iff $\xi^L = t^*$). L's favorite collective schedule, among those he can implement, is X^o: the others can all be constructed by beginning with X^o and then shifting resources from $[\xi, T)$ to $[t, \xi)$, where the latter all offer weakly higher δ^L-discounted marginal utility than the former. L thus chooses $\xi^L \in [t^o, \tau']$ in equilibrium, and the schedule implemented is x^o.

If $t^* < \tau' < t^o$, L implements the quasi-Stackelberg schedule with regime-change time ξ^L by choosing $\xi^L \in [t^*, \tau']$, and if L chooses $\xi^L \in (t, t^*)$, he implements either (a) the quasi-Stackelberg schedule with regime-change time τ' or (b) H's favorite feasible IC-polarized schedule that is open-loop after ξ^L. If he chooses $\xi^L \in (t, t^*)$ and H weakly prefers (b) to (a), then the regime-change time implemented by (b) cannot be after τ': if it were, then H would prefer to smooth her spending across $[t, \tau')$ rather than letting her spending jump down at ξ, as it must under (b). The collective schedule associated (b) can therefore be constructed by beginning from that associated with (a) and shifting resources from after to before ξ^L. The collective schedule associated with the quasi-Stackelberg schedule with regime-change time $\xi^L < \tau'$ can be constructed likewise. Since such shifts must lower L's payoff, L's favorite implementable schedule is the quasi-Stackelberg schedule with regime-change time τ'. This is the schedule implemented in equilibrium.

If $t^* \geq \tau'$, the equilibrium schedule is x^* regardless of ξ^L.

Suppose $\xi_\tau^H(h_{|t}) = \emptyset$ (and $\xi_\tau^L(h_{|t}) = t$). As shown in the previous subsection, on equilibrium schedules given the placement of the second announcement, the implemented schedule is IC-polarized for any choice of ξ^H. H's favorite feasible IC-polarized schedule is x^*. If $t^* < \tau'$, she can implement it by choosing $\xi^H = t^*$; if $t^* \geq \tau'$, she implements it with any choice of ξ^H. x^* is therefore always implemented in equilibrium.

Polarization of equilibrium schedules given placement of first announcement

Given a type 2 node $h_{|\tau}$, let $\xi^H \equiv \xi^H_\tau(h_{|\tau})$, and define ξ^L and $\hat{\xi}$ likewise. Let $x^o \equiv x^o[\tau, T, B^H_\tau, B^L_\tau]$ and $x^* \equiv x^*[\tau, T, B^H_\tau, B^L_\tau]$. We will show that any equilibrium schedule following $h_{|\tau}$ is IC-polarized.

If $\hat{\xi} = \xi^H = \xi^L = \tau'$, the equilibrium schedule(s) following $h_{|\tau}$ are those found following a type 5 node $h_{|t}$ (with τ in place of t), all of which are IC-polarized.

If $\hat{\xi} = \xi^H = \xi^L < \tau'$, the equilibrium schedule(s) following $h_{|\tau}$ are those found following a type 4 node $h_{|t}$ (with τ in place of t and $\hat{\xi}$ in place of $\hat{\xi}(h_{|t})$), all of which are IC-polarized.

If $\hat{\xi} = \xi^L < \xi^H$, the equilibrium schedule following $\hat{\xi}$ is Stackelberg. As we have seen, this implies that the equilibrium schedule following $h_{|\tau}$ is open-loop if $t^o \leq \hat{\xi}$, quasi-Stackelberg with regime-change time $\hat{\xi}$ if $t^* < \hat{\xi} < t^o$, and Stackelberg if $\hat{\xi} \leq t^*$. In every case, it is IC-polarized.

If $\hat{\xi} = \xi^H < \xi^L$, the equilibrium schedule following $\hat{\xi}$ is open-loop if $t^o[\hat{\xi}, T, B^H_{\hat{\xi}}, B^L_{\hat{\xi}}] \leq \tau'$, quasi-Stackelberg with regime-change time τ' if $t^*[\hat{\xi}, T, B^H_{\hat{\xi}}, B^L_{\hat{\xi}}] < \tau' < t^o[\hat{\xi}, T, B^H_{\hat{\xi}}, B^L_{\hat{\xi}}]$, and Stackelberg if $t^*[\hat{\xi}, T, B^H_{\hat{\xi}}, B^L_{\hat{\xi}}] \geq \tau'$. This is precisely analogous to the equilibrium schedules following a type 5 node $h_{|t}$ (with $\hat{\xi}$ in place of t). The equilibrium schedule(s) following $h_{|\tau}$ are those found following a type 4 node $h_{|t}$ (with τ in place of t and $\hat{\xi}$ in place of $\hat{\xi}(h_{|t})$), all of which are IC-polarized.

Unique equilibrium schedule from beginning of period

Given a type 1 node $h^-_{|\tau}$, let $x^o \equiv x^o[\tau, T, B^H_\tau, B^L_\tau]$ and $x^* \equiv x^*[\tau, T, B^H_\tau, B^L_\tau]$, let ξ^i denote $\sigma^{i*}(h^-_{|t})$, and let $h^+_{|\tau}$ denote the subsequent (type 2) node.

If $t^* \geq \tau'$, suppose $\xi^H = \tau'$. If $\xi^L = \tau'$, the unique equilibrium schedule following $h^+_{|\tau}$ is Stackelberg, as following a type 5 node with $t^* > \tau'$. If $\xi^L < \tau'$, the unique equilibrium schedule following ξ^L is Stackelberg; so, since $t^* > \xi^L$, the unique equilibrium schedule following $h^+_{|\tau}$ is also Stackelberg.

If $t^* < \tau'$, suppose $\xi^H = t^*$. If $\xi^L \geq t^*$, the unique equilibrium schedule following ξ^H is open-loop, quasi-Stackelberg, or Stackelberg, in the usual way; L does not spend before ξ^H in equilibrium, because $t^o > \xi^H$; and H can thus implement x^* by spending her budget δ^H-optimally across $[\tau, t^*)$. If $\xi^L < t^*$, the unique equilibrium schedule following ξ^L is Stackelberg; so, since $t^* > \xi^L$, the unique equilibrium schedule following $h^+_{|\tau}$ is also Stackelberg.

Since in equilibrium the schedule following $h^+_{|\tau}$ must be polarized, and since at $h^-_{|\tau}$ H can implement her favorite feasible IC-polarized schedule x^* with the appropriate strategy, x^* is the unique equilibrium schedule following $h^-_{|\tau}$.

A limit equilibrium schedule must be Stackelberg

For the rest of the proof, x^* and t^* will be as defined by Proposition 4.

Continuous time

Having completed the inductive step, we have proven that, fixing horizon T, the equilibrium schedule of game n is unique and Stackelberg. Denote this schedule by $x^{[T]}$. $x^{[T]}$ is independent of n, so it is also the unique equilibrium schedule in continuous time.

$x^{[T]}$ converges pointwise almost everywhere to x^* as $T \to \infty$

Let $U^H[\tilde{t}, T]$ denote H's payoff from a quasi-Stackelberg schedule with regime-change time $\tilde{t}$ across $[0, T)$. Let $f(\tilde{t}, T)$ denote the derivative of U^H with respect to $\tilde{t}$, as given by (102) with 0 replacing t, expressed as a function of $\tilde{t}$ and T. Observe that it is defined and continuous in both variables over $T \in (0, \infty]$ and $\tilde{t} \in (0, T)$.

Fixing B^H and B^L, let $t^*(T)$ denote the value of $\tilde{t}$ that sets (102) equal to 0 given horizon T. Recall that $t^*(T)$ is unique for all $T > 0$, including $T = \infty$. It can be found analytically that $t^*(\infty) = t^*$.

Fixing $\tilde{t}$, when T rises, Ⓑ$_{\tilde{t}}$ falls and Ⓒ$_{\tilde{t}}$ and Ⓓ$_{\tilde{t}}$ rise. By (105), as we have seen, this implies that $f(\tilde{t}, T)$ rises. It follows that $t^*(T)$ increases in T.

By the monotone convergence theorem, $\overline{t}^* \equiv \lim_{T\to\infty} t^*(T)$ is defined. Furthermore, because $f(\tilde{t}, T)$ is continuous, $f(\overline{t}^*, \infty) = \lim_{T\to\infty} f(t^*(T), T)$. This limit equals zero, since each of its elements equals zero by definition of $t^*(\cdot)$. Therefore $\overline{t}^* = t^*$.

It is then easy to verify that $x^{[T]}_t$ converges pointwise to x^*_t for all $t \neq t^*$.

A polarized equilibrium schedule must be Stackelberg

Let σ^* be a polarized equilibrium.

Let $h_{|t}$ be a post-announcement node. Suppose $\sigma^{L*}_s(h_{|t}) > 0$ for some s, and let $\underline{s} \equiv \inf\{s : \sigma^{L*}_s(h_{|t}) > 0\}$. (Recall that $\sigma^{L*}(h_{|t})$ is a spending plan for L across $[t, \hat{\xi}(h_{|t}))$, so $\underline{s} \in [t, \hat{\xi}(h_{|t})]$.) By definition of polarized equilibrium, $x^H_s(h_{|t}, \sigma^*) = 0$ for all $s \geq \underline{s}$ (the $s = \underline{s}$ case following from the assumption that spending plans are right-continuous). For σ^{L*} to be an optimal strategy for L, therefore, we must have $x^L_s(h_{|t}, \sigma^*) = B^L_{\underline{s}} e^{(r-\alpha^L)(s-\underline{s})}$ for all $s \geq \underline{s}$. Since σ^* is a polarized equilibrium and L begins spending before the next node, if $B^H(h_{|t}) > 0$, σ^{H*} must allocate $B^H(h_{|t})$ δ^H-optimally across $[t, \underline{s})$; and collective spending cannot be discontinuous at $\underline{s}$, or else restricting spending to before (weakly after) $\underline{s}$ cannot be a best response for H (L). So, if $B^H(h_{|t}) > 0$, we can only have $\sigma^{L*}(h_{|t}) \neq 0$ if the schedule implemented by σ^* following $h_{|t}$ is open-loop with a regime-change time before $\hat{\xi}(h_{|t})$. This in turn is possible only if $t^o[t, \infty, B^H(h_{|t}), B^L(h_{|t})] < \hat{\xi}(h_{|t})$:

$$\begin{aligned} & t^o[t, \infty, B^H(h_{|t}), B^L(h_{|t})] \geq \hat{\xi}(h_{|t}) \\ \Longrightarrow\ & \sigma^{L*}(h_{|t}) = 0 \qquad \text{for all post-announcement nodes } h_{|t}. \end{aligned} \tag{131}$$

Suppose that σ^H is such that, at any pre-announcement node $h_{|t}$ with $\xi^H_{\tau(t)}(h_{|t}) = \emptyset$ and $B^H(h_{|t}) > 0$, H chooses announcement

$$\sigma^H(h_{|t}) = \min(t^*[t, \infty, B^H(h_{|t}), B^L(h_{|t})], \tau'(t));$$

and that any post-announcement node $h_{|t}$ with $t^*[t, \infty, B^H(h_{|t}), B^L(h_{|t})] \geq \hat{\xi}(h_{|t})$, H chooses spending plan

$$\sigma^H(h_{|t}) = x^{H*}_{[t, \hat{\xi}(h_{|t}))}[t, \infty, B^H(h_{|t}), B^L(h_{|t})].$$

For any σ^{L*} satisfying (131), $x(\sigma^H, \sigma^{L*}) = x^*$. This can be seen by observing that (σ^H, σ^{L*}) implements x^* following all post-announcement nodes:

- At any type 2 node $h_{|t}$ with $B^H(h_{|t}) > 0$,

  $$\hat{\xi}(h_{|t}) \leq t^*[t, \infty, B^H(h_{|t}), B^L(h_{|t})] < t^o[t, \infty, B^H(h_{|t}), B^L(h_{|t})],$$

  so in equilibrium H is the only spender across $[t, \hat{\xi}(h_{|t}))$, and follows the Stackelberg schedule across this interval.

- H exhausts her budget at $\hat{\xi}(h_{|t})$ only if $\hat{\xi}(h_{|t}) = t^*[t, \infty, B^H(h_{|t}), B^L(h_{|t})]$. So at a type 4 node $h_{|t}$, either $B^H(h_{|t}) = 0$ and the Stackelberg schedule has been followed across $[\tau(t), t)$ (and will be implemented by L across $[t, \infty)$), or $B^H(h_{|t}) > 0$ and $\xi^H_{\tau(t)}(h_{|t}) = \emptyset$, in which case the next node will occur weakly before the Stackelberg regime-change time, and H will be the only spender until the next node, following the Stackelberg schedule.

- A type 5 node $h_{|t}$ can be reached only at the Stackelberg regime-change time, in which case $B^H(h_{|t}) = 0$, the Stackelberg schedule has been followed across $[\tau(t), t)$, and it will be implemented by L across $[t, \infty)$.

Since H can implement x^* when L employs any strategy σ^{L*} satisfying (131), since x^* is her favorite feasible IC-polarized schedule, and since $x(\sigma^*)$ is a feasible IC-polarized schedule for any polarized equilibrium σ^*, there is no polarized equilibrium σ^* with $x(\sigma^*) \neq x^*$.

A polarized equilibrium exists

Let σ^{H*} satisfy the conditions imposed on σ^H in the previous subsection, and define it fully by stipulating that, for a pre-announcement node $h_{|t}$ with $\xi^H_{\tau(t)}(h_{|t}) = \emptyset$ and $B^H(h_{|t}) = 0$, $\sigma^{H*}(h_{|t}) = \tau'(t)$, and for a post-announcement node $h_{|t}$ with $t^*[t, \infty, B^H(h_{|t}), B^L(h_{|t})] < \hat{\xi}(h_{|t})$, H allocates $B^H(h_{|t})$ δ^H-optimally across $[t, t^o[t, \infty, B^H(h_{|t}), B^L(h_{|t})])$ if $t^o \leq \tau'(t)$ and across $[t, \tau'(t)]$ otherwise.

Let σ^{L*} satisfy (131); for post-announcement nodes $h_{|t}$ with $t^o[t, \infty, B^H(h_{|t}), B^L(h_{|t})] < \hat{\xi}(h_{|t})$,

$$\sigma^{L*}(h_{|t}) = x^{L(o)}_{[t, \hat{\xi}(h_{|t}))}[t, \infty, B^H(h_{|t}), B^L(h_{|t})];$$

and for pre-announcement nodes $h_{|t}$ with $\xi^L_{\tau(t)}(h_{|t}) = \emptyset$, $\sigma^{L*}(h_{|t}) = \tau'(t)$.

Because σ^{L*} satisfies (131), and for any post-announcement node $h_{|t}$ with $t^o[t, \infty, B^H(h_{|t}), B^L(h_{|t})] < \tau'(t)$ we have $\sigma^{H*}_s(h_{|t}) = 0$ for all $s \geq t^o[t, \infty, B^H(h_{|t}), B^L(h_{|t})]$, $x_{[t,\infty)}(h_{|t}, \sigma^*)$ is polarized for all nodes $h_{|t}$. Therefore, if σ^* is an equilibrium, it is a polarized equilibrium.

Given that L adopts strategy σ^{L*}, $x((\sigma^H, \sigma^{L*}))$ is IC-polarized for any σ^H. Since $x(\sigma^*) = x^*$, which is H's favorite feasible IC-polarized schedule, σ^{H*} is a best response for H to σ^{L*}.

Following the node-cases of the "inductive step" above, it can be verified that for any grid point τ, if the players will play a strategy profile that implements the Stackelberg schedule from τ' onward (as σ^* does), and if H plays σ^{H*} across $[\tau, \tau')$, σ^{L*} is a best response for L for every possible node that might arise across $[\tau, \tau')$. The proofs of L's best responses through the inductive step rely nowhere on the assumption that T is finite.

When $\gamma < 1$, infinite-horizon game n is continuous at infinity (for any n), since the range of feasible payoffs at any period is bounded above and below and the stream of flow payoffs cannot grow more quickly than the players' discount rates, by (9). By the one-shot deviation principle, therefore, σ^{L*} is a best response for L to σ^{H*}. This establishes that σ^* is an equilibrium of the infinite-horizon game when $\gamma < 1$.

When $\gamma \geq 1$, however, the game is not continuous at infinity. We must verify explicitly that, given that H plays σ^{H*}, at any node $h_{|t}$ L has no incentive to deviate permanently from σ^{L*} to an alternative strategy σ^L.

σ^{L*} maximizes L's forward-looking optimization problem at all nodes $h_{|t}$ with $B^H(h_{|t}) = 0$. Deviation to σ^L therefore can only be profitable for L at a node $h_{|t}$ with $B^H(h_{|t}) > 0$. Suppose by contradiction that there is such an $h_{|t}$ and σ^L.

If $\sigma \equiv (\sigma^{H*}, \sigma^L)$ is such that

$$\exists \tau \in G^{(n)} : B^H(h) = 0, \quad h \equiv h^-_{|\tau}(h_{|t}, \sigma), \tag{132}$$

then σ^L offers L a lower payoff than σ^{L*} at $h_{|t}$. This can be seen by backward induction. First, because of the unique optimality for L of the schedule implemented by σ^{L*} from h onward, a permanent deviation to σ^L cannot offer L higher utility than a deviation to σ^L until h followed by a reversion to σ^{L*}, and must offer L lower utility if $x(h,\sigma) \neq x(h,\sigma^*)$. Then, given that the players play σ^* from τ onward, and given that H plays σ^{H*} from $\underline{\tau}$ onward (where $\underline{\tau}$ denotes the grid point before τ), it follows from the "inductive step" of the proof of the unique limit equilibrium that any strategy $\tilde{\sigma}^L$ among L's most preferred strategies at any node $h_{|s}$ with $s \in [\underline{\tau}, \tau)$ must satisfy $x(h_{|s}, \tilde{\sigma}) = x(h_{|s}, \sigma^*)$. By induction, deviation to a strategy σ^L satisfying (132) is weakly undesirable for L at any node.

Consider a strategy $\tilde{\sigma}^L$ not satisfying (132). We will assume by contradiction that permanently deviating to $\tilde{\sigma}^L$ at $h_{|t}$ increases L's payoff, and show that this implies that L's payoff to playing $\tilde{\sigma}^L$ from time τ onward is $-\infty$ for some grid point $\tau > t$.

By the reasoning above, for any grid point $\tau > t$, L's payoff to playing $\tilde{\sigma}^L$ at all nodes $h_{|s}$ with $s \in [t,\tau)$, and σ^{L*} from τ onward, is weakly less than his payoff to playing σ^{L*} from $h_{|t}$ onward. So, in general denoting L's continuation payoff to playing strategy σ^L from grid point $\tau > t$ onward after playing $\tilde{\sigma}^L$ at all nodes $h_{|s}$ with $s \in [t,\tau)$ by

$$C(\sigma^L,\tau) \equiv \int_\tau^\infty e^{-\delta^L(s-\tau))} u\big(X_s\big(h^-_{|\tau}(h_{|t},\tilde{\sigma}),(\sigma^{H*},\sigma^L)\big)\big)ds,$$

and denoting L's payoff to playing σ^L from $h_{|t}$ onward by $C(\sigma^L,t)$, we have

$$\begin{aligned} C(\sigma^{L*},t) &\geq C(\tilde{\sigma}^L,t) + e^{-\delta^L(\tau-t)}\big(C(\sigma^{L*},\tau) - C(\tilde{\sigma}^L,\tau)\big) \\ \implies C(\tilde{\sigma}^L,\tau) - C(\sigma^{L*},\tau) &\geq e^{\delta^L(\tau-t)}\big(C(\tilde{\sigma}^L,t) - C(\sigma^{L*},t)\big). \end{aligned} \tag{133}$$

If $\tilde{\sigma}^L$ is a profitable deviation for L at $h_{|t}$, the right-hand side of (133) is positive, so the difference in continuation payoffs as a function of $\tau - t$ must be "fast-growing", which we will define to mean asymptotically bounded below by $c_0 e^{\delta^L(\tau-t)}$ for some constant $c_0 > 0$. $C(\tilde{\sigma}^L,\tau)$ can never exceed the continuation payoff for L at τ obtained if both parties invest all funds from t to τ and subsequently disburse them δ^L-optimally. This continuation payoff plateaus if $\gamma > 1$, and grows linearly in $\tau - t$ at absolute rate r if $\gamma = 1$. (See the payoff expression from Proposition 1, substituting $B(h_{|t})e^{r(\tau-t)}$ for B.) For the difference in continuation payoffs to be fast-growing, therefore, $C(\sigma^{L*},\tau)$ must eventually be negative and its absolute value fast-growing.

Since σ^* implements the Stackelberg schedule from any pre-announcement grid node onward, $C(\sigma^{L*},\tau) = U^L(x^*[\tau])$, where $x^*[\tau] \equiv x^*[\tau,\infty,B^H(h^-_{|\tau}(h_{|t},\tilde{\sigma})),B^L(h^-_{|\tau}(h_{|t},\tilde{\sigma}))]$. An expression for this payoff can be found analytically by integrating

$$\int_\tau^\infty e^{-\delta^L(s-\tau)} u(x^*_s[\tau])ds,$$

because the regime-change point $t^*[\tau]$ can be found analytically in this setting where the horizon is infinite (see Proposition 4). The result can be written in terms of the collective initial budget B_τ and L's initial budget share b_τ as

$$\frac{B_\tau^{1-\gamma}}{1-\gamma} \cdot \frac{\left(\alpha^L\right)^{-\gamma}}{\alpha^H\alpha^L\gamma + (\delta^H-\delta^L)^2(1-\gamma)} \cdot \Big(\alpha^H\alpha^L\gamma\big(b_\tau + (1-b_\tau)\frac{\alpha^H}{\alpha^L}\eta\big)^{1-\gamma} + (\delta^H-\delta^L)^2 \\ (1-\gamma) b_\tau^{\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H}} \big((b_\tau + (1-b_\tau)\frac{\alpha^H}{\alpha^L}\eta\big)^{-\frac{\alpha^L}{\alpha^H}\gamma}\Big), \quad \gamma > 1; \tag{134}$$

$$\frac{1}{\delta^L}\Big(\frac{(\delta^H-\delta^L)^2}{\delta^H\delta^L}\Big(1 + \frac{(1-b_\tau)\delta^H}{b_\tau\delta^L}\eta\Big)^{-\frac{\delta^L}{\delta^H}} + \ln(B_\tau) + \ln\big(b_\tau\delta^L + (1-b_\tau)\delta^H\eta\big) \\ + \frac{\delta^H-\delta^L}{\delta^H} + \frac{r-\delta^H}{\delta^L}\Big), \qquad \gamma = 1.$$

If $\gamma > 1$, the coefficient on $B_\tau^{1-\gamma}$ is negative and is bounded below across $b_\tau \in [0,1]$. For $C(\sigma^{L*}, \tau)$ to be fast-growing, therefore, $B_\tau^{1-\gamma}$ must be fast-growing. B_τ must thus eventually be bounded above by $c_1 e^{\frac{\delta^L}{1-\gamma}(\tau-t)}$ for some $c_1 > 0$. Because the spending rate cannot sustainably shrink more slowly than the collective budget, the discounted continuation payoff to adopting σ^L from a sufficiently large grid point $\tau > t$ must likewise be bounded above, for some $c_2 > 0$, by

$$\int_\tau^\infty e^{-\delta^L(s-t)} \frac{\big(c_2 e^{\frac{\delta^L}{1-\gamma}s}\big)^{1-\gamma}}{1-\gamma} ds = -\infty.$$

If $\gamma = 1$, the terms added to $\ln(B_\tau)$ are likewise bounded across $b_\tau \in [0,1]$. For $C(\sigma^{L*}, \tau)$ to be fast-growing, therefore, $\ln(B_\tau)$ must be (again, negative and) fast-growing, so B_τ must eventually be bounded above by a function $f(\tau)$ that falls superexponentially to zero quickly enough that $\ln(f(\tau))$ is (negative and) fast-growing. Again, because the spending rate cannot sustainably shrink more slowly than the collective budget, the discounted continuation payoff to following σ^L from sufficiently large τ must be bounded above, for some $c_1, c_2 > 0$, by

$$\int_\tau^\infty e^{-\delta^L(s-t)} \ln\big(c_1 f(s)^{c_2}\big) ds = -\infty.$$

This contradicts the assumption that σ^L is a profitable deviation at $h_{|t}$ from σ^{L*}, whose payoff is well-defined and finite (as confirmed by (134), recalling conditions (9) and (18)). σ^* is thus an equilibrium of infinite-horizon game n for all n.

A.6 Proof of Proposition 6

Preliminaries

Following the notation of Appendix A.5, we will let $x^*[t, T, B^H, B^L]$ denote the Stackelberg schedule beginning at t with budgets B^H and B^L and ending at T, dropping some or all of the arguments where clear. X^*, etc. are defined likewise. We will also let $b^H \equiv B^H/B = 1 - b$.

We begin by showing that the frontier of efficient payoffs is strictly concave.

Let $U^0 = (U^{H(0)}, U^{L(0)})$ and $U^1 = (U^{H(1)}, U^{L(1)})$ be two efficient payoff profiles, and let $X^{(0)}$ and $X^{(1)}$ be collective schedules which attain these payoff profiles.

The mixture collective schedule $X^{(\zeta)}$, defined by $X_t^{(\zeta)} = \zeta X_t^{(1)} + (1-\zeta)X_t^{(0)}$, is feasible:

$$\begin{aligned}
&\int_0^\infty e^{-rt}\big(\zeta X_t^{(1)} + (1-\zeta)X_t^{(0)}\big)dt \\
&= \zeta \int_0^\infty e^{-rt}X_t^{(1)}dt + (1-\zeta)\int_0^\infty e^{-rt}X_t^{(0)}dt \\
&= \zeta B + (1-\zeta)B = B.
\end{aligned}$$

Furthermore, for $\zeta \in (0,1)$, the discounted flow utility that $X^{(\zeta)}$ offers player i at each time t is defined and equal to $e^{-\delta^i t}u(\zeta X_t^{(1)} + (1-\zeta)X_t^{(0)})$. By the strict concavity of $u(\cdot)$, this is strictly greater than the ζ-mixture of the discounted flow utilities offered by $X^{(0)}$ and $X^{(1)}$, i.e. $e^{-\delta_i t}(\zeta u(X_t^{(1)}) + (1-\zeta)u(X_t^{(0)}))$.

Thus X^ζ offers a payoff profile that is weakly Pareto-superior to $\zeta U^1 + (1-\zeta)U^0$. It follows that the frontier of efficient payoffs cannot exhibit any (even weak) convexities.

Because the frontier of efficient payoffs is concave, an efficient collective schedule X must maximize

$$U^a(X) \equiv aU^H(X) + (1-a)U^L(X)$$

for some payoff weight $a \in [0,1]$. In particular, given efficient collective schedule X, the corresponding U^a cannot be increased by shifting resources between time 0 and any other time t. Given that $X_{(\cdot)}$ is right-continuous, this implies

$$X_0^{-\gamma} = e^{rt}\big(ae^{-\delta^H t} + (1-a)e^{-\delta^L t}\big)X_t^{-\gamma} \quad \forall t. \tag{135}$$

So X is optimal according to time preference factor

$$\beta_t^{(a)} = ae^{-\delta^H t} + (1-a)e^{-\delta^L t}, \tag{136}$$

or time preference rate

$$\delta_t^{(a)} = -\frac{\dot{\beta}_t^{(a)}}{\beta_t^{(a)}} = \frac{a\delta^H e^{-\delta^H t} + (1-a)\delta^L e^{-\delta^L t}}{ae^{-\delta^H t} + (1-a)e^{-\delta^L t}}. \tag{137}$$

As we can see, $\delta_0^{(a)} = a\delta^H + (1-a)\delta^L$. Therefore a is not only the weight placed on H's payoff, but also the weight placed on her time preference in determining the starting time preference rate.

Given payoff weight a, let $w_t^{(a)}$ denote the weight placed on H's time preference rate at t, such that

$$\delta_t^{(a)} = w_t^{(a)}\delta^H + (1 - w_t^{(a)})\delta^L. \tag{138}$$

Substituting (137) into (138) and rearranging, we have

$$w_t^{(a)} = \frac{a}{a + (1-a)e^{(\delta^H - \delta^L)t}}.$$

Observe that

$$w_0^{(a)} = a. \tag{139}$$

Fixing payoff weight a, the resulting schedule is not time-consistent, because the resulting time preference rate (137) is not constant. Upon reaching time $s > 0$, $aU^H + (1-a)U^L$ is maximized across times $t \geq s$ by allocating the collective budget according to time preference rate schedule δ_{t-s}, as prescribed, not according to a fixed δ_s.

However, if upon reaching s we instead place weight

$$\tilde{a} \equiv w_s^{(a)}$$

on H's future payoff (and $1 - \tilde{a}$ on L's), the resulting time preference rate schedule is the same across $t \geq s$ as that prescribed at time 0 using payoff weight a. That is,

$$\delta_{t-s}^{(\tilde{a})} = \delta_t^{(a)} \quad \forall s \geq t.$$

We can see this by substituting $w_s^{(a)}$ for a into (136), simplifying, and differentiating:

$$\begin{aligned}
\beta_{t-s}^{(\tilde{a})} &= \frac{a \cdot e^{-\delta^H (t-s)}}{a + (1-a)e^{(\delta^H - \delta^L)s}} + \frac{(1-a)e^{(\delta^H - \delta^L)s} \cdot e^{-\delta^L (t-s)}}{a + (1-a)e^{(\delta^H - \delta^L)s}} \\
&= \frac{e^{\delta^H s}}{a + (1-a)e^{(\delta^H - \delta^L)s}} \left(ae^{-\delta^H t} + (1-a)e^{-\delta^L t}\right) \\
\implies \delta_{t-s}^{(\tilde{a})} &= -\frac{\dot{\beta}_{t-s}^{(\tilde{a})}}{\beta_{t-s}^{(\tilde{a})}} = \frac{a\delta^H e^{-\delta^H t} + (1-a)\delta^L e^{-\delta^L t}}{ae^{-\delta^H t} + (1-a)e^{-\delta^L t}} = \delta_t^{(a)}.
\end{aligned}$$

Let $X^{(a)}$ now denote the efficient collective schedule implied by payoff weight a. It is unique by the strict concavity of the feasible payoff set. It is also $\mathcal{C}^1$ in a, as can be seeing by rearranging and integrating (135) to get

$$\int_0^\infty e^{\frac{r-r\gamma}{\gamma}} X_0^{(a)} \frac{1}{ae^{-\delta^H t} + (1-a)e^{-\delta^L t}} dt - B = 0 \tag{140}$$

and applying the implicit function theorem to find that $X_0^{(a)}$, and thus $X_t^{(a)}$ for each t, is $\mathcal{C}^1$ in a.

Let $\chi^{(a)}$ denote the corresponding collective schedule defined by

$$\chi_t^{(a)} \equiv X_t^{(a)}/B.$$

Likewise, for $b^H \in [0,1]$, let $\chi^*[b^H]$ denote the collective schedule defined by

$$\chi_t^*[b^H] \equiv X_t^*[Bb^H, B(1-b^H)]/B.$$

Given b^H, by the homotheticity of $U(\cdot)$, $X^{(a)}$ is [weakly] Pareto superior to X^* iff $\chi^{(a)}$ is [weakly] Pareto superior to $\chi^*[b^H]$.

Given any $\chi^{(a)}$ for $a \in (0,1)$, there is an interval $(\underline{b}^H, \overline{b}^H) \subset (0,1)$ such that $\chi^{(a)}$ is a strict Pareto improvement on $\chi^*[b^H]$ iff $b^H \in (\underline{b}^H, \overline{b}^H)$. This follows from the inefficiency of $\chi^*[b^H]$ for $b^H \in (0,1)$ and the facts that

i. $U\big(\chi^*[0]\big) = U\big(\chi^{(0)}\big)$,

ii. $U\big(\chi^*[1]\big) = U\big(\chi^{(1)}\big)$,

iii. $U^H\big(\chi^*[b^H]\big)$ is continuous and strictly increasing (in fact $\mathcal{C}^1$ with a positive derivative) in b^H, and

iv. $U^L\big(\chi^*[b^H]\big)$ is continuous and strictly decreasing (in fact $\mathcal{C}^1$ with a negative derivative) in b^H.

We can thus define

$$\underline{b}^H(a) \equiv \underset{b^H}{\text{argmin}} : U^L\big(\chi^{(a)}\big) \geq U^L\big(\chi^*[b^H]\big),$$

$$\overline{b}^H(a) \equiv \underset{b^H}{\text{argmax}} : U^H\big(\chi^{(a)}\big) \geq U^H\big(\chi^*[b^H]\big),$$

with $\overline{b}^H(a) > \underline{b}^H(a)$. Recall that $X^{(a)}$, and thus $U^i(\chi^{(a)})$, are $\mathcal{C}^1$ in a. By the implicit function theorem and (iii)-(iv), $\underline{b}^H(a)$ and $\overline{b}^H(a)$ are also $\mathcal{C}^1$ in a, with negative derivatives.

By construction, $(\underline{b}^H(a), \bar{b}^H(a))$ is the range of initial impatient budget shares b^H such that $\chi^{(a)}$ is strictly Pareto superior to $\chi^*[b^H]$. As noted above, by homotheticity of $U(\cdot)$, it is also the b^H-range such that $X^{(a)}$ is Pareto superior to X^*. So both parties weakly prefer $X^{(a)}$ to X^* at $t = 0$ iff

$$b^H \in \big(\underline{b}^H(a), \bar{b}^H(a)\big). \tag{141}$$

More generally, given a schedule x, it holds at all t that both parties weakly prefer the forward-looking collective schedule $X^{(a)}_{[t,\infty)}$ to the forward-looking Stackelberg collective schedule $X^*[t, \infty, B^H(x_{|t}), B^L(x_{|t})]$ iff

$$b^H(x_{|t}) \in \big(\underline{b}^H\big(w_t^{(a)}\big), \bar{b}^H\big(w_t^{(a)}\big)\big) \ \ \forall t. \tag{142}$$

Observe that $\frac{\partial}{\partial t} w_t^{(a)} < 0$ and thus that $\frac{\partial}{\partial t}\underline{b}^H\big(w_t^{(a)}\big) < 0$ and $\frac{\partial}{\partial t}\bar{b}^H\big(w_t^{(a)}\big) < 0$.

We will now show that, for any a satisfying (141), there is a schedule x such that $X = X^{(a)}$ and (142) holds.

A schedule everywhere superior to forward-looking Stackelberg

Observe that $\underline{b}^H\big(w_t^{(a)}\big)$ and $\bar{b}^H\big(w_t^{(a)}\big)$ are $\mathcal{C}^1$ in t for any payoff weight a. We can therefore define a differentiable path of budget shares for H, $\{\tilde{b}_t^H\}$, by

$$\tilde{b}_0^H = b^H;$$

$$\frac{d}{dt}\tilde{b}_t^H = \max\Big(0, \frac{1+p-2p_t}{1-p}\Big) \max\Big(f_1(\tilde{b}_t^H, t), f_2(\tilde{b}_t^H, t)\Big) + \min\Big(2\frac{p_t - p}{1-p}, 1\Big) f_2(\tilde{b}_t^H, t), \tag{143}$$

where

$$f_1(\tilde{b}_t^H, t) \equiv \frac{(\tilde{b}_t^H - \underline{b}^H)\big(\frac{d}{dt}\bar{b}^H(w_t^{(a)}) - \frac{d}{dt}\underline{b}^H(w_t^{(a)})\big)}{\bar{b}^H(w_t^{(a)}) - \underline{b}^H(w_t^{(a)})}, \quad f_2(\tilde{b}_t^H, t) \equiv (\tilde{b}_t^H - 1)\frac{X_t^{(a)}}{B(X_{|t}^{(a)})},$$

$$p \equiv \frac{b^H - \underline{b}^H(a)}{\bar{b}^H(a) - \underline{b}^H(a)} \in (0,1), \qquad p_t \equiv \frac{\tilde{b}_t^H - \underline{b}^H\big(w_t^{(a)}\big)}{\bar{b}^H\big(w_t^{(a)}\big) - \underline{b}^H\big(w_t^{(a)}\big)}.$$

Because the right-hand side of (143) is continuous in t (fixing $\tilde{b}_t^H$) and Lipschitz continuous in $\tilde{b}_t^H$ across $[0, T]$ for all T, $\tilde{b}_T^H$ is defined and $\mathcal{C}^1$ for all T by the Picard–Lindelöf theorem.

Recall that $p_0 = p$ by (139), and observe that $p_{(\cdot)}$ is differentiable. We will show that $p_t \in (0,1)$ for all t.

First, observe that $\frac{d}{dt}\tilde{b}_t^H$ linearly interpolates, in p_t, from $\max(f_1, f_2)$ at $p_t \leq p$ to f_2 at $p_t \geq \frac{1+p}{2}$. Also, $\frac{d}{dt}p_t$ strictly increases in $\frac{d}{dt}\tilde{b}_t^H$, and when $\frac{d}{dt}\tilde{b}_t^H = f_1(\tilde{b}_t^H, t)$, $\frac{d}{dt}p_t = 0$. So if $p_t \leq p$, $\frac{d}{dt}p_t \geq 0$. This establishes that we never reach a t with $p_t < p$.

Next, suppose by contradiction that $p_{t^*} = 1$ for some t^*. Let t^* denote the minimum such time, noting that a minimum exists because $p_{(\cdot)}$ is continuous and $[0,\infty)$ is closed below. Then $\tilde{b}^H_{t^*}/\overline{b}^H\big(w^{(a)}_{t^*}\big) = 1$. Since p_t is continuous and $p_0 < 1$, there is an $s \in (0,t^*)$ such that $p_t \in [\frac{1+p}{2},1)\ \forall t \in [s,t^*]$, and therefore also such that $\tilde{b}^H_t/\overline{b}^H\big(w^{(a)}_t\big) < 1$ throughout this interval. However,

$$\frac{d}{dt}\frac{\tilde{b}^H_t}{\overline{b}^H(w^{(a)}_t)} \leq 0 \ \ \forall t \in [s,t^*].$$

This is because the proportional, and thus the absolute, growth rate of $\tilde{b}^H(x_{|t})/\overline{b}^H(w^{(a)}_t)$ is non-positive iff

$$\frac{\frac{d}{dt}\overline{b}^H\big(w^{(a)}_t\big)}{\overline{b}^H\big(w^{(a)}_t\big)} \geq \frac{\frac{d}{dt}b^H(x_{|t})}{b^H(x_{|t})}.$$

By (143) and the fact that $\tilde{b}^H(x_{|t}) \leq \overline{b}^H(w^{(a)}_t)$ for $t \in [s,t^*]$,

$$\frac{\frac{d}{dt}\tilde{b}^H(x_{|t})}{\tilde{b}^H(x_{|t})} = \frac{\big(\tilde{b}^H(x_{|t})-1\big)X^{(a)}_t/B(X^{(a)}_{|t})}{b^H(x_{|t})} \leq \frac{\big(\overline{b}^H\big(w^{(a)}_t\big)-1\big)X^{(a)}_t/B(X^{(a)}_{|t})}{\overline{b}^H\big(w^{(a)}_t\big)} \ \ \forall t \in [s,t^*].$$

The proof is thus completed by showing that

$$\frac{\frac{d}{dt}\overline{b}^H\big(w^{(a)}_t\big)}{\overline{b}^H\big(w^{(a)}_t\big)} \geq \frac{\big(\overline{b}^H(w^{(a)}_t)-1\big)X^{(a)}_t/B(X^{(a)}_{|t})}{\overline{b}^H(w^{(a)}_t)} \ \ \forall t \in [s,t^*]. \tag{144}$$

Suppose by contradiction that (144) fails for some $t \in [s,t^*]$. The right-hand side equals $(\frac{d}{dt}b^H_t)/b^H_t$ given $b^H_t = \overline{b}^H(w^{(a)}_t)$ and $x^H_t = X^{(a)}_t$. But $b^H_t = \overline{b}^H(w^{(a)}_t)$ implies that H is indifferent between the forward-looking efficient and Stackelberg schedules:

$$U^H\big(X^{(a)}_{[t,\infty)}\big) = U^H\big(X^*[t,\infty,B^H(x_{|t}),B^L(x_{|t})]\big). \tag{145}$$

$b^H_t > 0$ and the failure of (144) then imply that there is an $\epsilon > 0$ such that it is feasible to set $x^H_s = X^{(a)}_s$ for $s \in [t,t+\epsilon)$, and such that if this obtains, $\overline{b}^H$ falls by a larger proportion than b^H across the interval. We then have $\underline{b}^H\big(w^{(a)}_{t+\epsilon}\big) < b^H_{t+\epsilon}$ and thus

$$U^H\big(X^{(a)}_{[t+\epsilon,\infty)}\big) < U^H\big(X^*[t+\epsilon,\infty,B^H(x_{|t+\epsilon}),B^L(x_{|t+\epsilon})]\big) \implies$$
$$U^H\big(X^{(a)}_{[t,\infty)}\big) < U^H\big(X^{(a)}_{[t,t+\epsilon)}\big) + e^{-\delta^H\epsilon}U^H\big(X^*[t+\epsilon,\infty,B^H(x_{|t+\epsilon}),B^L(x_{|t+\epsilon})]\big). \tag{146}$$

But this is impossible, since the schedule across the right-hand side of (146)—$x_s^H = X_s^{(a)}, x_s^L = 0$ for $s \in [t, t+\epsilon)$ followed by a Stackelberg schedule—is an IC-polarized schedule from t onward, and H's favorite such schedule is the Stackelberg schedule itself from t onward (see Appendix A.5, "Stackelberg schedules", the $\bar{t} = \infty$ case), whose payoff for H equals the left-hand side of (146) by (145). So (144) holds.

So, given the path of impatient budget shares $\{\tilde{b}_t^H\}$ characterized by (143), $p_t \in (0,1)\ \forall t$.

We will now show that there is a unique schedule $x^{(a)}$ such that $x^{H(a)} + x^{L(a)} = X^{(a)}$ and $b^H(x_{|t}^{(a)}) = \tilde{b}_t^H\ \forall t$, and confirm that $x^{(a)}$ is (i) feasible in the sense that $x_t^{i(a)} \geq 0\ \forall i, t$ and (ii) continuous. We will not need to further verify that $x^{(a)}$ does not violate either player's budget constraint, since we have already established that $\tilde{b}_t^H \in (0,1)\ \forall t$.

Given a schedule $x^{(a)}$ with $x^{H(a)} + x^{L(a)} = X^{(a)}$, $b^H(x_{|t}^{(a)}) = \tilde{b}_t^H\ \forall t$ iff

$$\frac{d}{dt}\tilde{b}_t^H = \frac{d}{dt}b^H(x_{|t}^{(a)}) = \frac{\tilde{b}_t^H X_t^{(a)} - x_t^{H(a)}}{B(X_{|t}^{(a)})}$$

$$\implies \frac{x_t^{H(a)}}{X_t^{(a)}} = \tilde{b}_t^H - \frac{d}{dt}\tilde{b}_t^H \cdot \frac{B(X_{|t}^{(a)})}{X_t^{(a)}} \tag{147}$$

(with $x_t^{L(a)}/X_t^{(a)} = 1 - x_t^{H(a)}/X_t^{(a)}$).

$x^{(a)}$ is thus well defined. Furthermore (147) > 1 at t iff $\frac{d}{dt}\tilde{b}_t^H < f_2(\tilde{b}_t^H, t)$, and (147) < 0 at t only if $\frac{d}{dt}\tilde{b}_t^H < 0$. Both are impossible, by (143) and the non-positivity of f_1 and f_2. So $x^{(a)}$ is feasible. Finally, because $\tilde{b}_t^H$ is $\mathcal{C}^1$ in t, $x^{(a)}$ is continuous.

This completes the construction of a feasible schedule $x^{(a)}$ that is continuous and satisfies (142).

Constructing a grid and equilibrium

Given a grid G and a payoff weight a satisfying (141), let σ^* be a polarized equilibrium, and let $\tilde{\sigma}$ be the strategy profile of the game played on G with

$$\tilde{\sigma}^i(h_{|t}) = \begin{cases} \tau'(t), & h_{|t} = h_{|t}^-(\tilde{\sigma}); \\ x_{[t,\tau'(t))}^{i(a)}, & h_{|t} = h_{|t}^+(\tilde{\sigma}), \\ \sigma^{i*}(h_{|t}) & \text{otherwise.} \end{cases} \tag{148}$$

Note that any node reached on the path of $\tilde{\sigma}$ must occur at a grid point $t \in G$, and so be of type 1 or type 2.

We will now construct a grid G such that σ^* is an equilibrium of the game played on G.

Let $\tau_0 = 0$, and define τ_k for $k > 1$ recursively as follows.

Let $\hat{X}[t,\epsilon]$ be the truncated schedule across $[t,\infty)$ with

$$\hat{X}_s[t,\epsilon] \equiv X^*_{t+\epsilon}[t+\epsilon,\infty,\hat{B}^H[t,\epsilon],\hat{B}^H[t,\epsilon]]e^{(r-\alpha^H)(s-(t+\epsilon))}, \quad s \in [t,t+\epsilon);$$
$$X^*[t+\epsilon,\infty,\hat{B}^H[t,\epsilon],\hat{B}^L[t,\epsilon]], \qquad\qquad s \geq t+\epsilon,$$

where

$$\hat{B}^i[t,\epsilon] \equiv \Big(B^i\big(x^{(a)}_{|t}\big) + \int_t^{t+\epsilon} e^{-r(s-t)}x^{-i(a)}_s ds\Big)e^{r\epsilon} \;\; \forall i.$$

To interpret $\hat{X}[t,\epsilon]$, suppose that $x_{|t} = x^{(a)}_{|t}$. Suppose then that, at t, each i receives a budget-increase equal to the present value of the resources allocated by $-i$ across $[t,t+\epsilon)$ according to $x^{-i(a)}$; invests the entirety of the newly increased B^i_t until $t+\epsilon$ (so that their budget at $t+\epsilon$ is then $\hat{B}^i[t,\epsilon]$); and subsequently implements the Stackelberg schedule. Then

$$X_{t+\epsilon} = X^*_{t+\epsilon}[t+\epsilon,\infty,\hat{B}^H[t,\epsilon],\hat{B}^L[t,\epsilon]].$$

$\hat{X}_{[t,t+\epsilon)}[t,\epsilon]$ is then the truncated collective schedule obtained across $[t,t+\epsilon)$ by positing that collective spending grows at rate $r-\alpha^H$ across $[t,t+\epsilon)$ (as it does just after $t+\epsilon$) and is continuous at $t+\epsilon$. $\hat{X}[t,\epsilon]$ is of course infeasible given initial budgets $\{B^i(x^{(a)}_{|t})\}$, for any $\epsilon > 0$. In particular, observe that for all $s \geq t$, $\hat{X}_s[t,0] = X^*_s[t,\infty,B^H(x^{(a)}_t),B^L(x^{(a)}_t)]$, and $\hat{X}_s[t,\epsilon]$ is $\mathcal{C}^1$ in ϵ with a derivative strictly positive for all $\epsilon \geq 0$. Observe also that $\hat{X}$ is Stackelberg with respect to the present-value budgets it allocates from player.

Then let

$$\epsilon^i(t) \equiv \epsilon > 0 : U^i\big(X^{(a)}_{[t,\infty)}\big) = U^i\big(\hat{X}[t,\epsilon]\big),$$
$$\epsilon(t) \equiv \min(\epsilon^H(t),\epsilon^L(t)).$$

$\epsilon^i(t)$ exists and is unique for each i because

$$U^i\big(\hat{X}[t,0]\big) = U^i\big(X^*[t,\infty,B^H(x^{(a)}_{|t}),B^L(x^{(a)}_{|t})]\big) < U^i\big(X^{(a)}_{[t,\infty)}\big);$$

$\lim_{\epsilon\to\infty} U^i(\hat{X}[t,\epsilon])$ is the supremum feasible payoff (∞ if $\gamma \leq 1$, 0 if $\gamma > 1$); and $U^i(\hat{X}[t,\epsilon])$ is continuous and monotonic in ϵ.

Let $\tau_k = \tau_{k-1} + \epsilon(\tau_{k-1})$.

We must establish that $\lim_{k\to\infty}\tau_k = \infty$, so that the grid is locally finite and the game is well-defined. This follows from the facts that, if not, then $\lim_{k\to\infty}\tau_k = \overline{\tau}$ for some $\overline{\tau} < \infty$, by the monotone convergence theorem, and thus $\epsilon^i(\tau_k) \to 0$ for some i; that $\epsilon^i(t)$ is continuous in t for each i, by the implicit function theorem; and $\epsilon^i(\overline{\tau}) > 0$ for each i.

To establish that $\tilde{\sigma}$ is an equilibrium of the game played on grid $G = \{\tau_k\}_{k\in\mathbb{N}}$, we will show that no player profits by deviating at any node. Given a node τ, we will let $\hat{X} \equiv \hat{X}[\tau, \tau' - \tau]$.

If $h_{|t}$ is off the path of $\tilde{\sigma}$, this follows from the fact that σ^* is an SPE.

If $h_{|\tau}$ is on path and of type 1, a deviation by H induces an IC-polarized subsequent schedule, which she must weakly disprefer to $X^*[t, \infty, B^H_\tau, B^L_\tau]$ (see Appendix A.5); and since $X^*_s[t, \infty, B^H_\tau, B^L_\tau] < \hat{X}_s \ \forall s \geq \tau$, she prefers the latter schedule. Since a deviation by L to an announcement $\xi^L < \tau'$ will be followed by polarized equilibrium behavior from τ onward, it follows from Appendix A.5: Inductive step: Polarization of equilibrium schedules given placement of first announcement (the $\hat{\xi} = \xi^H < \xi^L$ case) that L weakly prefers the schedule implemented by σ^* given $\xi^L = \tau'$ to that implemented by σ^* given any other ξ^L. This schedule likewise exhibits lower spending than $\hat{X}$ across $[\tau, \infty)$ in all three cases (open-loop, quasi-Stackelberg, and Stackelberg).

If $h_{|\tau}$ is on path and of type 2, first observe that

$$X^*_{\tau'}[\tau', \infty, B^H(x^{(a)}_{|\tau'}), B^L(x^{(a)}_{|\tau'})] > X^{(a)}_{\tau'}. \tag{149}$$

If (149) failed, then, because the growth rate of $X^{(a)}$ is everywhere within $(r - \alpha^H, r - \alpha^L)$ by (136), and because $X^{(a)}$ cannot lie everywhere above X^* after τ' given the same initial budgets since both collective schedules are budget-exhausting, there would have to be a $\bar{t} > t^*[\tau', \infty, B^H(x^{(a)}_{|\tau'}), B^L(x^{(a)}_{|\tau'})]$ such that

$$X^*_t[\tau', \infty, B^H(x^{(a)}_{|\tau'}), B^L(x^{(a)}_{|\tau'})] < / = / > X^{(a)}_t, \quad t < / = / > \bar{t}.$$

$X^{(a)}$ could thus be constructed by shifting resources from times after $\bar{t}$ to times before $\bar{t}$. Since all of the latter offer weakly lower δ^L-discounted marginal utility than all of the former, $X^{(a)}_{[\tau',\infty)}$ is then not a Pareto improvement on $X^*[\tau', \infty, B^H(x^{(a)}_{|\tau'}), B^L(x^{(a)}_{|\tau'})]$, as we know it to be by construction.

Next, observe that if X is a collective schedule following τ that i can implement by deviating at $h_{|\tau}$, i can also implement X given

$$B^i_\tau = \tilde{B}^i_\tau \equiv B^i(x^{(a)}_{|\tau}) + \int_\tau^{\tau'} e^{-r(t-\tau)} x^{-i(a)}_t dt,$$

$$B^{-i}_\tau = \tilde{B}^{-i}_\tau \equiv B^{-i}(x^{(a)}_{|\tau}) - \int_\tau^{\tau'} e^{-r(t-\tau)} x^{-i(a)}_t dt,$$

$$x^{-i}_t = 0, \quad t \in [\tau, \tau')$$

(and polarized equilibrium behavior from τ' onward) by reserving the resources "transferred" from the other player for spending before τ' as prescribed by $x^{-i(a)}$.

Under these circumstances, with $i = H$, any schedule H implements is an IC-polarized allocation of $\{\tilde{B}^i\}$. It follows from (149) that

$$\tilde{B}^H < \int_\tau^\infty e^{-r(t-\tau)} \hat{X}_t^H dt,$$

and so the same also holds for L. To summarize, therefore, H prefers $\hat{X}$ to $X^*[\tau, \infty, \tilde{B}^H, \tilde{B}^L]$, as both are Stackelberg but the former uses larger budgets for both players (and $U^H(X^*[B^H, B^L])$ is increasing in both arguments); weakly prefers $X^*[\tau, \infty, \tilde{B}^H, \tilde{B}^L]$ to any IC-polarized allocation of $\{\tilde{B}^i\}$; and weakly prefers an IC-polarized allocation of $\{\tilde{B}^i\}$ to any schedule she can implement by deviating at τ.

With $i = L$, he maximizes his payoff by setting a δ^L-optimal schedule across $[\tau, \tau')$ with some fraction of his $\tilde{B}_\tau^L$ and allocating the rest to the Stackelberg schedule that will obtain following τ'. (A maximal payoff exists by the extreme value theorem, since U^L is continuous in the fraction of $\tilde{B}_\tau^L$ allocated before τ'.) Whatever fraction he chooses, the resulting schedule following τ' is Stackelberg with an initial budget for H of $\tilde{B}^H e^{r(\tau'-\tau)} < \hat{B}^H$ and an initial budget for L weakly less than $\hat{B}^L$ (with equality only if L spends nothing before τ').

Since $U^L(X^*[B^H, B^L])$ increases in both arguments (see Appendix A.5: Final period: Unique equilibrium schedule given placement of first announcement), an optimal deviation by L produces a schedule X with

$$U^L(X_{[\tau',\infty)}) < U^L(\hat{X}_{[\tau',\infty)}).$$

The cited subsection also establishes that an optimal division of $\tilde{B}_\tau^L$ across τ' must yield $\lim_{t\to\tau'^+} X_t \leq X_{\tau'}$, and that in general $X_t^*[t, \infty, B_t^H, B_t^L]$ increases in B_t^i for both i. Since $\hat{B}^i$ must be weakly greater than $B^i(x_{|\tau'})$ for both i, $\lim_{t\to\tau'^+} X_t \leq \hat{X}_{\tau'}$. Finally, since to be δ^L-optimal X must grow at rate $r - \alpha^L$ across $[\tau, \tau')$, and since $\hat{X}$ grows at the slower rate $r - \alpha^H$ across this interval, $\hat{X}_s > X_s \; \forall s \in [\tau, \tau')$ and thus

$$U^L(X_{[\tau,\tau')}) < U^L(\hat{X}_{[\tau,\tau')}).$$

We have shown that σ^* is an equilibrium on any grid G with $\tau' \leq \tau + \epsilon(\tau) \; \forall \tau \in G$.

To find an admissible grid sequence such that σ^* is an equilibrium of every grid in the sequence, let $f(t) \equiv \{f_k(t)\}_{k\in\mathbb{N}}$ be the sequence defined recursively by

$$f_0(t) = t, \quad f_k(t) = f_{k-1}(t) + \epsilon(f_{k-1}(t)) \;\; (k > 0), \tag{150}$$

so that $G = f(0)$. Then let $G^{(0)} = G$, and let

$$G^{(n)} = G^{(n-1)} \cup \bigcup_{m=0}^{\infty} f\left(\frac{m}{2^{n-1}}\right), \quad n > 0.$$

Intermediate equilibrium payoffs

So far we have established that any payoff profile $\overline{U}$ that is (i) efficient and (ii) strictly Pareto superior to $U(X^*)$ is an equilibrium payoff profile. We will now show that any feasible payoff profile U that is Pareto superior to $U(X^*)$ is an equilibrium payoff profile.

Let $[\underline{a}, \overline{a}]$ be the range of values of a such that $X^{(a)}$ is Pareto superior to X^*. Observe that, since $\chi_t^{(a)}$ is continuous in a for all t and nowhere constant in a (see (140)), $U^i\big(X^{(a)}\big)$ is continuous and strictly monotonic in a for each i.

Define $f^{(a)}(t)$ as in (150), but with $\epsilon(\cdot)$ defined with respect to a.

Suppose U is efficient but only weakly Pareto superior to $U(X^*)$, with $U^i = U^i(X^*)$. Let

$$a^i \in \{\underline{a}, \overline{a}\} \equiv a : U^i\big(X^{(a)}\big) = U^i\big(X^*\big),$$

and recall that for any $a \in (\underline{a}, \overline{a})$, there a grid G and a strategy profile σ such that σ is an equilibrium of the game given grid G and $X(\sigma) = X^{(a)}$.

We can therefore define a monotonic sequence $\{a_n\}$ with $a_0 \in (\underline{a}, \overline{a})$ and $a_n \to a^i$, and a corresponding grid sequence with $G^{(0)} = f^{(a_0)}(0)$ and

$$G^{(n)} = \bigcup_{k=0}^{\infty} f^{(a_n)}\big(\tau_k^{(n-1)}\big) \cup \bigcup_{m=0}^{\infty} f^{(a_n)}\Big(\frac{m}{2^{n-1}}\Big), \quad n > 0.$$

The first union ensures that $G^{(n)} \supset G^{(n-1)}$. The second ensures that $\cup_{n=0}^{\infty} G^{(n)}$ is dense. The inclusion of all elements of $f^{(a_n)}(\tau)$ for $\tau \in G^{(n)}$ ensures that, for each n, there is a strategy profile $\sigma^{(n)}$ that is an equilibrium of the game played on $G^{(n)}$ and such that $X(\sigma^{(n)}) = X^{(a_n)}$. Since $X_t^{(a_n)} \to X_t^{(a^i)}$ for all t, $X^{(a^i)}$ is an equilibrium collective schedule. Since $U(X^{(a_n)}) \to U = U(X^{(a^i)})$, U is an equilibrium payoff.

Suppose U is Pareto superior to $U(X^*)$ but inefficient. Then there is an efficient payoff profile $\overline{U}$ strictly Pareto superior to U (and so also to $U(X^*)$) and a $C \in (0,1)$ such that

$$U = \overline{U}[C] \equiv \begin{cases} C^{1-\gamma}\overline{U}, & \gamma \neq 1; \\ \big(\overline{U}^H + \ln(C), \overline{U}^L + \ln(C)\big), & \gamma = 1. \end{cases}$$

Let $a \in (0,1)$ be such that $U(X^{(a)}) = \overline{U}$.

Let σ^* be a polarized equilibrium, and for $c \in (C, 1)$, let $\sigma^{(c)}$ be the strategy profile with

$$\sigma^i(h_{|t}) = \begin{cases} \tau'(t), & h_{|t} = h_{|t}^-(\sigma); \\ \big\{\frac{B^i}{\tau_1}(1-c)e^{rs}\big\}_{s\in[0,\tau_1)}, & t = 0 \text{ and } h_{|t} = h_{|t}^+(\sigma); \\ ce^{r\tau_1} x^{i(a)}_{[t-\tau_1, \tau'(t)-\tau_1)}, & t > 0 \text{ and } h_{|t} = h_{|t}^+(\sigma); \\ \sigma^{i*}(h_{|t}), & \text{otherwise.} \end{cases}$$

Given $\sigma^{(c)}$, at time 0, each i chooses announcement $\xi^i = \tau_1$ and then allocates fraction $1-c$ of B^i uniformly across $[0, \tau_1)$. The players then play $\tilde{\sigma}$, as defined by (148), with initial time τ_1 in place of 0 and initial budgets $\{ce^{r\tau_1}B^i\}$ in place of $\{B^i\}$. We will now find a τ_1 such that $\sigma^{(c)}$ is an equilibrium of the game played on grid $G \equiv 0 \cup \big(\tau_1 + f^{(a)}(0)\big)$, where $\tau_1 + f^{(a)}(0)$ denotes the set $f^{(a)}(0)$ with τ_1 added to each element. Note that $f^{(a)}(0)$ constructs a grid on which $\tilde{\sigma}$ is an equilibrium given any pair of budgets in the same ratio as $\{B^i\}$, as $\{ce^{r\tau_1}B^i\}$ is.

It follows from the fact that $\tilde{\sigma}$ and σ^* are SPEs that σ is an equilibrium at nodes after time 0. To verify that σ is an equilibrium at 0 for sufficiently small τ_1, i's payoff to defecting at either the type 1 or the type 2 node at $t=0$ can be upper-bounded by the payoff i receives if the entire budget is allocated δ^i-optimally across $[0, \tau_1)$ and also (of course infeasibly) invested to τ_1 and subsequently Stackelberg:

$$\begin{aligned}
&\int_0^{\tau_1} e^{-\delta^i t} u\Big(\frac{B\alpha^i}{1-e^{-\alpha^i \tau_1}} e^{(r-\alpha^i)t}\Big) dt + e^{-\delta^i \tau_1} U^i\big(X^*[\tau_1, \infty, B^H e^{r\tau_1}, B^L e^{r\tau_1}]\big) \\
&\to 0 + U^i\big(X^*[0, \infty, B^H, B^L]\big) = U^i(X^*)
\end{aligned}$$

as $\tau_1 \to 0^+$. $U^i(X(\sigma^{(c)}))$ converges to $\overline{U}^i[c]$ as $\tau_1 \to 0^+$. Since $c > C$, $\overline{U}^i[c] > \overline{U}^i[C] = U^i \geq U^i(X^*)$.

Given $c \in (C, 1)$, let $\tau_1[c]$ be a time such that $\sigma^{(c)}$ is an equilibrium on any grid $G = 0 \cup \big(\tau_1 + f^{(a)}(0)\big)$ with $\tau_1 \leq \tau_1[c]$. Let $\{c_n\}$ be a decreasing sequence with $c_0 \in (C, 1)$ and $c_n \to C$. We can then define a corresponding grid sequence with $G^{(0)} = 0 \cup \big(\tau_1[c_0] + f^{(a)}(0)\big)$ and

$$G^{(n)} = G^{(n-1)} \cup \bigcup_{k=0}^{\infty} \big(\tau_1^{(n)} + f^{(a)}\big(\tau_k^{(n-1)} - \tau_1^{(n)}\big)\big) \cup \bigcup_{m=0}^{\infty} f^{(a)}\Big(\frac{m}{2^{n-1}}\Big), \quad n > 0,$$

where $\tau_1^{(n)} = \min(1/2^{n-1}, \tau_1[c_n])$. The first term ensures that $G^{(n)} \supset G^{(n-1)}$. The second ensures that $\sigma^{(c_n)}$ is an equilibrium of the on-path subgame beginning at $\tau_1^{(n)}$. The third again ensures that $\cup_{n=0}^{\infty} G^{(n)}$ is dense.

$\{\sigma^{(c_n)}\}$ is thus a sequence of equilibria across an admissible grid sequence. Since $X_t(\sigma^{(c_n)}) \to CX_t^{(a)}$ $\forall t \neq 0$ (since $\tau_1^{(n)} \to 0$), $CX^{(a)}$ is an equilibrium collective schedule. Since $U\big(X(\sigma^{(c_n)})\big) \to U\big(CX^{(a)}\big) = U$, U is an equilibrium payoff.

A.7 Proof of Proposition 7

Existence and uniqueness of constrained-polarized equilibrium schedule

Let σ^* be a strategy profile in which, at each post-announcement node $h_{|t}$, for $s \in [t, \xi(h_{|t}))$ we have

$$\sigma_s^{H*}(h_{|t}) = B^H(h_{|t})\overline{\alpha}e^{(r-\overline{\alpha})(s-t)}, \tag{151}$$

$$\sigma_s^{L*}(h_{|t}) = \begin{cases} 0, & s < \bar{t}^*(h_{|t}); \\ \big(B^H(h_{|t})e^{(r-\overline{\alpha})(\bar{t}^*(h_{|t})-t)} + B^L(h_{|t})e^{r(\bar{t}^*(h_{|t})-t)}\big) & \\ \quad \cdot \alpha^L e^{(r-\alpha^L)(s-\bar{t}^*(h_{|t}))} - B^H(h_{|t})\overline{\alpha}e^{(r-\overline{\alpha})(s-t)}, & s \geq \bar{t}^*(h_{|t}), \end{cases} \tag{152}$$

where

$$\bar{t}^*(h_{|t}) \equiv t + \max\left(0, \ln\left(\frac{B^H(h_{|t})}{B^L(h_{|t})}\frac{\overline{\alpha}-\alpha^L}{\alpha^L}\right)\Big/\overline{\alpha}\right). \tag{153}$$

Observe that

i. by (151), $x^H((\sigma^{H*},\sigma^L))$ is independent of σ^L;

ii. by (151) and (152), $X_s(\sigma^*)$ is the δ^L-optimal allocation of $B_{\bar{t}^*(h_{|t})}$ across $s \geq \bar{t}^*(h_{|t})$; and

iii. by (152) and (153), $x^L_{\bar{t}^*(h_{|t})}(\sigma^*) = 0$ if $\bar{t}^*(h_{|t}) > t$.

By (ii), $x^L(\sigma^*)$ exhausts L's budget in present-value terms. By (i) and (iii), any alternative schedule for L either would sub-optimally allocate his resources weakly after $\bar{t}^*(h_{|t})$ or would shift some of L's resources from weakly after $\bar{t}^*(h_{|t})$ to times before $\bar{t}^*(h_{|t})$ which, because $\alpha^L \leq \overline{\alpha}$ and X is continuous at $\bar{t}^*(h_{|t})$, offer weakly lower δ^L-discounted marginal utility. It follows that σ^{L*} is a best response to σ^{H*}, and that for any constrained-polarized equilibrium σ, $x(\sigma) = x(\sigma^*)$.

As with the unconstrained game of Proposition 5, the constrained game is continuous at infinity iff $\gamma < 1$. In this case, to prove that σ^{H*} is a best response to σ^{L*}, we can use the one-shot deviation principle. Let $h_{|t}$ be a post-announcement node and let $\xi \equiv \xi(h_{|t})$.

Let $Y_s \equiv e^{-rs}B_t^H$ denote the present value, as of time 0, of H's budget at s, and let it be written as a function of the history and/or strategy profile as B_t^H can. $\dot{Y}_s \leq 0$, since H has no outside income, and $\dot{Y}_s \geq \overline{\alpha}Y_s$ due to the spending constraint. Since x^H is right-continuous, any deviation by H at $h_{|t}$ to a strategy $\tilde{\sigma}^H$ with $\tilde{\sigma}_s^H(h_{|t}) < B_s^H\overline{\alpha}$ for some $s \in [t,\xi)$ raises $\dot{Y}$ from its lower bound for an interval following s, which raises Y_ξ and thus B_ξ^H. So a deviation by H shifts resources from some times in $[t,\xi)$ (a) to increases in B_ξ^H and possibly also (b) to later times in the $[t,\xi)$ interval. We will now show that both shifts lower H's payoff.

Regarding (b): let

$$\overline{\mathcal{T}} \equiv \{s \in [t,\xi) : \tilde{\sigma}_s^H(h_{|t}) > X_s(h_{|t},\sigma^*)\}, \tag{154}$$

$$\underline{\mathcal{T}} \equiv \{s \in [t,\xi) : \tilde{\sigma}_s^H(h_{|t}) < X_s(h_{|t},\sigma^*)\}$$

denote the set of times in $[t, \xi)$ at which H spends more (less) under σ^H than under σ^{H*}. Let

$$\overline{Y} \equiv \int_{\overline{T}} e^{-rs}\Big(\tilde{\sigma}_s^H(h_{|t}) - B^H(h_{|t})e^{(r-\overline{\alpha})(s-t)}\Big)ds.$$

Let

$$t^{-1}(y) \equiv s : Y_t - Y_s(h_{|t}, \sigma^*) = y$$

$$\tilde{t}^{-1}(y) \equiv s : Y_t - Y_s(h_{|t}, (\tilde{\sigma}^H, \sigma^{L*})) = y.$$

Observe first that $t^{-1}(y)$ is defined for all but a countable number of $y \in [Y(h_{|t}), Y_\xi(h_{|t}, \sigma^*))$. If a value of y in this interval is undefined, there exist $t_1, t_2 > t_1$ such that

$$Y_{t_1}(h_{|t}, \sigma^*) = Y_{t_2}(h_{|t}, \sigma^*) = y,$$

so, by the monotonicity of $Y_{(\cdot)}$, $Y_s = y$ for all $s \in [t_1, t_2]$. Thus if there were an uncountable number of values of y in the interval for which $t^{-1}(y)$ were undefined, the interval would contain an uncountable number of disjoint intervals, which it does not (as can be seen e.g. from the fact that each interval contains a distinct rational, and the rationals are countable). Likewise, $\tilde{t}^{-1}(y)$ is defined almost everywhere for $(\tilde{\sigma}^H, \sigma^{L*})$. Now observe that, for any $y \in [0, \overline{Y}]$ for which $t^{-1}(y)$ and $\tilde{t}^{-1}(y)$ are defined,

$$\tilde{t}^{-1}(y) \geq t^{-1}(y).$$

Finally, the payoff gains for H achieved by any above-baseline expenditures before ξ implemented by $\tilde{\sigma}^H$ are bounded above by

$$\int_0^{\overline{Y}} e^{(r-\delta^H)\tilde{t}^{-1}(y)} u'\Big(X_{\tilde{t}^{-1}(y)}(h_{|t}, \sigma^*)\Big)dy, \tag{155}$$

and the payoff losses to the first $\overline{Y}$ units of below-baseline expenditures are bounded below by

$$\int_0^{\overline{Y}} e^{(r-\delta^H)t^{-1}(y)} u'\Big(X_{\tilde{t}^{-1}(y)}(h_{|t}, \sigma^*)\Big)dy. \tag{156}$$

(The integrals are defined because $t^{-1}(\cdot)$ and $\tilde{t}^{-1}(\cdot)$ are defined almost everywhere.) Since $\tilde{t}^{-1}(y) \geq t^{-1}(y)$ almost everywhere, and since $X_{(\cdot)}(h_{|t}, \sigma^*)$ always grows at a proportional rate weakly greater than $r - \alpha^H$, the expression in integral (155) is less than or equal to that in (156) for every value of y. So any reallocations within $[t, \xi)$ implemented by $\tilde{\sigma}^H$ from the σ^{H*} baseline cannot raise H's payoff.

Regarding (a): we will show that increases to B_ξ^H increase X_s for all $s \geq \xi$. Let $\tilde{h} \equiv h_\xi^+(h_{|t}, (\tilde{\sigma}^H, \sigma^{L*}))$ and $h^* \equiv h_\xi^+(h_{|t}, \sigma^*)$. Observe that $\overline{t}^*(h)$ is continuous and weakly increasing in $B^H(h)$, and that $x^H_s(h,\sigma^*)$ is proportional to $B^H(h)$ for all $s \geq \xi$.

If $\overline{t}^*(\tilde{h}) = t$, then $\overline{t}^*(h^*) = t$; the schedule implemented by σ^* from ξ onward is in either case the δ^L-optimal allocation of the collective budget B_ξ; and the increase to B^H_ξ proportionally increases X_s for all $s \geq \xi$.

If $\overline{t}^*(\tilde{h}) > t$, then the increase to B^H_ξ increases $\overline{t}^*$—i.e. $\overline{t}^*(\tilde{h}) > \overline{t}^*(h^*)$—unless $\overline{\alpha} = \alpha^L$ (in which case $\overline{t}^*(\tilde{h}) = \overline{t}^*(h^*) = \xi$ regardless of B^H_ξ). And $X_{\overline{t}^*(\tilde{h})}(\tilde{h},\sigma^*) > X_{\overline{t}^*(\tilde{h})}(h^*,\sigma^*)$: if it were not, then the collective resources allocated across $[\overline{t}^*(\tilde{h}),\infty)$ would be weakly less following $\tilde{h}$ than following h^*, even while the resources allocated by H would be strictly greater, so L's spending (now beginning at $\overline{t}^*(\tilde{h}) \geq \overline{t}^*(h^*)$) would not exhaust his budget. Since $r - \overline{\alpha} \leq r - \alpha^L$, $X_s(\tilde{h},\sigma^*) > X_s(h^*,\sigma^*)$ for all $s \in [\xi, \overline{t}^*(\tilde{h}))$. So again, the increase to B^H_ξ increases X_s for all $s \geq \xi$.

This proves that one-shot deviations do not raise H's payoff. If $\gamma < 1$, this completes the proof that σ^{H*} is a best response to σ^{L*}.

Suppose $\gamma \geq 1$. Given that post-announcement node $h_{|t}$ has been reached, denote H's continuation payoff to playing strategy σ^H from grid point $\tau > t$ onward after playing $\tilde{\sigma}^L$ at all nodes $h_{|s}$ with $s \in [t,\tau)$ by

$$C(\sigma^H,\tau) \equiv \int_\tau^\infty e^{-\delta^H(s-\tau))} u\big(X_s\big(h^-_{|\tau}(h_{|t},\tilde{\sigma}),(\sigma^H,\sigma^{L*})\big)\big)ds,$$

and denote H's payoff to playing σ^H from $h_{|t}$ onward by $C(\sigma^H,t)$. By backward induction on one-shot deviations, deviation from σ^{H*} to $\tilde{\sigma}^H$ for only a finite length of time lowers H's payoff following $h_{|t}$. We therefore have

$$\begin{aligned} & C(\sigma^{H*},t) > C(\tilde{\sigma}^H,t) + e^{-\delta^H(\tau-t)}\big(C(\sigma^{H*},\tau) - C(\tilde{\sigma}^H,\tau)\big) \\ \implies & C(\tilde{\sigma}^H,\tau) - C(\sigma^{H*},\tau) > e^{\delta^H(\tau-t)}\big(C(\tilde{\sigma}^H,t) - C(\sigma^{H*},t)\big). \end{aligned}$$

If $\tilde{\sigma}^H$ is a profitable deviation for H at $h_{|t}$, the right-hand side of (133) is positive, so the difference in continuation payoffs as a function of $\tau - t$ must be "fast-growing", as defined to mean asymptotically bounded below by $c_0 e^{\delta^H(\tau-t)}$ for some constant $c_0 > 0$. $C(\tilde{\sigma}^H,\tau)$ can never exceed the continuation payoff for H at τ obtained if both parties invest all funds from t to τ and subsequently disburse them δ^H-optimally. (This is in fact a very loose upper bound, since it ignores the feasibility constraint imposed by the spending maximum.) This continuation payoff plateaus if $\gamma > 1$, and grows linearly in $\tau - t$ at absolute rate r if $\gamma = 1$. (See the payoff expression from Proposition 1, substituting $B(h_{|t})e^{r(\tau-t)}$ for B.) For the difference in continuation payoffs to be fast-growing, therefore, $C(\sigma^{H*},\tau)$ must eventually be negative and its absolute value fast-growing.

But this is impossible, since $C(\sigma^{H*},\tau)$ is bounded below by H's continuation payoff at τ in the event that she minimizes B^H_τ by spending as quickly as possible until τ—so that

$$B^H_\tau = B^H_t e^{(r-\overline{\alpha})(\tau-t)}$$

—and then enjoys only $X_s = x^H_s(h_{|\tau}, \sigma^*)$ for $s \geq \tau$, so that we ignore contributions from L. By the homotheticity of $U(\cdot)$, this lower bound is proportional to $u(B^H_\tau)$. Thus if $\gamma = 1$, this lower bound is linear in τ, and if $\gamma > 1$, this lower bound is proportional to $-e^{(r-\overline{\alpha})(1-\gamma)\tau}$. By $\overline{\alpha} < \alpha^H$ and (9), $r < \overline{\alpha} \implies (r-\overline{\alpha})(1-\gamma) < \delta^H$.

So σ^{H*} is a best response to σ^{L*} given any value of γ.

Open-loop schedule

By (i), the proof that σ^{L*} is a best response to σ^{H*} also serves as a proof that, in the open-loop setting, $\overline{x}^L$ is a best response to $\overline{x}^H$. The proof that, given σ^{L*}, H weakly disprefers reallocations from σ^{H*} (at a node $h_{|t}$) implemented within the $[t, \xi)$ interval also, with $t = 0$ and $\xi = \infty$, serves as a proof that, in the open-loop setting, $\overline{x}^H$ is a best response to $\overline{x}^L$.

A.8 Proof of Proposition 8

Existence and uniqueness of constrained-polarized equilibrium schedule

Let σ^* be a strategy profile in which, at each post-announcement node $h_{|t}$, for $s \in [t, \xi(h_{|t}))$ we have

$$\sigma^{L*}_s(h_{|t}) = B^L(h_{|t})\underline{\alpha}e^{(r-\underline{\alpha})(s-t)}, \tag{157}$$

$$\sigma^{H*}_s(h_{|t}) = \begin{cases} \left(B^L(h_{|t})e^{(r-\underline{\alpha})(\underline{t}^*(h_{|t})-t)}\right)\underline{\alpha}e^{(r-\alpha^H)(s-\underline{t}^*(h_{|t}))} \\ \quad -B^L(h_{|t})\underline{\alpha}e^{(r-\underline{\alpha})(s-t)}, & s < \underline{t}^*(h_{|t}), \\ 0, & s \geq \underline{t}^*(h_{|t}); \end{cases} \tag{158}$$

where $\underline{t}^*(h_{|t})$ uniquely satisfies

$$\alpha^H/b(h_{|t}) = \underline{\alpha}e^{(\alpha^H-\underline{\alpha})(\underline{t}^*-t)} + (\alpha^H - \underline{\alpha})e^{-\underline{\alpha}(\underline{t}^*-t)} \tag{159}$$

(or equals ∞ if $B^L(h_{|t}) = b(h_{|t}) = 0$, or if $\underline{\alpha} = \alpha^H$).

To verify that there is a $\underline{t}^*(h_{|t}) \geq t$ satisfying (159) if $b(h_{|t}) > 0$ and $\underline{\alpha} < \alpha^H$, observe that the right-hand side of (159) equals α^H at $\underline{t}^* = t$ and that, given $\alpha^H > \underline{\alpha}$, it rises without bound as $\underline{t}^* \to \infty$. To verify that $\underline{t}^*(h_{|t})$ is unique, observe that the derivative of the right-hand side of (159) equals 0 at $\underline{t}^* = t$ and is positive at $\underline{t}^* > t$.

Equality (159), and $\underline{t}^*(h_{|t}) = \infty$ given $b(h_{|t}) = 0$ or $\underline{\alpha} = \alpha^H$, is derived by setting

$$B^H(h_{|t}) = \int_t^{\underline{t}^*(h_{|t})} e^{-r(s-t)}\sigma^{H*}_s(h_{|t})ds,$$

so that H precisely exhausts her budget at $\underline{t}^*(h_{|t})$.

Observe that

i. by (157), $x^L((\sigma^H, \sigma^{L*}))$ is independent of σ^H;

ii. by (158), σ^* implements the δ^H-optimal allocation of resources spent across $[t, \underline{t}^*(h_{|t}))$; and

iii. by (158), $\lim_{s \to \underline{t}^{*-}(h_{|t})} x_t^H(\sigma^*) = 0$.

So any alternative schedule for H either would sub-optimally allocate her resources weakly before $\underline{t}^*(h_{|t})$ or would shift some of her resources from weakly before $\underline{t}^*(h_{|t})$ to times on or after $\underline{t}^*(h_{|t})$ which, because $\alpha^H \geq \underline{\alpha}$ and X is continuous at $\underline{t}^*(h_{|t})$, offer weakly lower δ^H-discounted marginal utility. It follows that σ^{H*} is a best response to σ^{L*}, and that for any constrained-polarized equilibrium σ, $x(\sigma) = x(\sigma^*)$.

As with the unconstrained game of Proposition 5, the constrained game is continuous at infinity iff $\gamma < 1$. In this case, to prove that σ^{L*} is a best response to σ^{H*}, we can use the one-shot deviation principle. Let $h_{|t}$ be a post-announcement node and let $\xi \equiv \xi(h_{|t})$. By a proof precisely analogous to that in the case of a spending minimum (see Appendix A.8), deviation by L at $h_{|t}$ to a strategy implementing an alternative path of x^L until ξ must shift resources from all times on and after ξ to some times before ξ, and may also shift resources from later to earlier periods within $[t, \xi)$; and both effects lower L's payoff. If $\gamma < 1$, this completes the proof that σ^{L*} is a best response to σ^{H*}.

If $\gamma \geq 1$, the proof that σ^{L*} is a best response to σ^{H*} is, for the first two paragraphs, identical to the proof that σ^{H*} is a best response to σ^{L*} in the case of a spending minimum (where σ^* is defined as in Appendix A.8), with the L and H indices reversed. Here, to disprove that $C(\sigma^{L*}, \tau)$ is negative and fast-growing, let

$$B_\tau^i \equiv B^i(x_{|\tau}(h_{|t}, (\sigma^{H*}, \tilde{\sigma}^L))),$$

and observe that $C(\sigma^{L*}, \tau)$ is bounded below both by

- L's continuation payoff at τ if he allocates B_τ^L δ^L-optimally from τ onward in the absence of H, which is given by (32), and by

- L's continuation payoff at τ if H allocates B_τ^H δ^H-optimally from τ onward in the absence of L, which is given by $U_{\delta^L}(B_\tau^H, \alpha^H)$ as defined by (33).

As in the proof that a polarized equilibrium exists in the unconstrained setting (Appendix 5), this in turn implies for any $\gamma \geq 1$ that B_τ^L and B_τ^H (and thus also B_τ) are asymptotically bounded above by an expression that declines to zero quickly enough that $\lim_{\tau \to \infty} e^{-\delta^L(\tau - t)} C(\tilde{\sigma}^L, \tau)$ cannot feasibly exceed $-\infty$. This contradicts the assumption that $\tilde{\sigma}^L$ is a profitable deviation from σ^{L*}.

Open-loop schedule

By (i), the proof that σ^{H*} is a best response to σ^{L*} also serves as a proof that, in the open-loop setting, $\overline{x}^H$ is a best response to $\overline{x}^L$. The proof that, given σ^{H*}, L weakly disprefers reallocations from σ^{L*} (at a node $h_{|t}$) implemented within the $[t, \xi)$ interval also, with $t = 0$ and $\xi = \infty$, serves as a proof that, in the open-loop setting, $\overline{x}^L$ is a best response to $\overline{x}^H$.

A.9 Proof of Proposition 9

Existence and uniqueness of constrained-polarized equilibrium schedule

Suppose that, at each post-announcement node $h_{|t}$, the players' spending plans follow

$$\underline{\overline{x}}^H_s = B^H(h_{|t})\overline{\alpha}e^{(r-\overline{\alpha})(s-t)},$$
$$\underline{\overline{x}}^L_s = B^L(h_{|t})\underline{\alpha}e^{(r-\underline{\alpha})(s-t)}.$$

Denote these spending plans $\{\underline{\overline{x}}^i(h_{|t})\}$.

Any feasible deviation by H from to a strategy implementing $x^H \neq \underline{\overline{x}}^H(h_{|t})$, following any node, shifts H's resources from earlier to later times—in the sense of (154)–(156) and the surrounding discussion—without affecting $x^L = \underline{\overline{x}}^L(h_{|t})$. From the $\underline{\overline{X}}(h_{|t})$ baseline, increases in allocations to later times offer lower δ^H-discounted marginal utility than increases in allocations to earlier times—

$$s_2 > s_1 \implies e^{-(r-\delta^H)(s_2-t)}u'\big(\underline{\overline{X}}_{s_1}(h_{|t})\big) \leq e^{-(r-\delta^H)(s_1-t)}u'\big(\underline{\overline{X}}_{s_2}(h_{|t})\big)$$

—since the growth rate of $\underline{\overline{X}}$ is everywhere a weighted average of $r - \underline{\alpha}$ and $r - \overline{\alpha}$, which weakly exceeds $r - \alpha^H$. The deviation thus lowers H's payoff.

Fixing $x^H = \underline{\overline{x}}^H(h_{|t})$, any feasible deviation by L likewise only shifts resources from later to earlier times and, since a weighted average of $r - \underline{\alpha}$ and $r - \overline{\alpha}$ is everywhere weakly less $r - \alpha^L$, lowers L's payoff.

Given existence, uniqueness here follows immediately from the definition of constrained polarization.

Open-loop schedule

Since the above applies to any $h_{|t}$, including $h_{|t} = \emptyset$, $\underline{\overline{x}}$ is also an equilibrium of the open-loop game.

A.10 Proof of Proposition 10

Assume throughout that spending minima and maxima satisfy (23).

Given a spending rate $\underline{\alpha}$ ($\overline{\alpha}$), define $\underline{\delta}$ ($\overline{\delta}$) as the time preference rate for which the corresponding spending rate is optimal, as in (34).

Bounded WTP for H given both constraints

H's payoff given a spending minimum $\underline{\alpha} \geq \alpha^L$ only, $U^H(x[B^H, B^L, \underline{\alpha}, \infty])$, is less than

$$U_{\delta^H}(B^L, \underline{\alpha}) + \left(B^L \underline{\alpha}\right)^{-\gamma} B^H.$$

This is the payoff she receives from L spending B^L at rate $\underline{\alpha}$, as he does under $\underline{x}^L$, plus the payoff-increase implied if she could spend all of B^H at time zero, when the δ^H-discounted marginal utility to resource allocations is highest from the $\underline{x}^L$ baseline, without confronting the diminishing marginal utility to her own spending. Then substituting $B^H(1-w)$ for B^H and using the explicit expression for $U_{\delta^H}(\cdot)$ from (33), we have

$$\begin{aligned} U^H\left(x[B^H(1-w), B^L, \underline{\alpha}, \infty]\right) &< U_{\delta^H}(B^L, \underline{\alpha}) + \left(B^L \underline{\alpha}\right)^{-\gamma} B^H(1-w) \\ &= \frac{\left(B^L\right)^{1-\gamma}}{1-\gamma} \frac{\underline{\alpha}^{1-\gamma}}{\underline{\alpha} + \delta^H - \underline{\delta}} + \left(B^L \underline{\alpha}\right)^{-\gamma} B^H(1-w), \quad \gamma \neq 1; \quad (160) \\ &= \frac{\delta^H \ln\left(B^L \underline{\delta}\right) + r - \underline{\delta}}{\delta^{H2}} + \left(B^L \underline{\delta}\right)^{-1} B^H(1-w), \quad \gamma = 1. \end{aligned}$$

H's payoff given spending minimum $\underline{\alpha} \geq \alpha^L$ and spending maximum $\overline{\alpha} \leq \alpha^H$, $U^H(x[B^H, B^L, \underline{\alpha}, \overline{\alpha}])$, is weakly greater than $U_{\delta^H}(B, \underline{\alpha})$—the payoff she receives when the collective budget B is spent at rate $\underline{\alpha}$—since given $\overline{x}^L$, she can achieve this payoff by choosing $x_t^H = B^H \underline{\alpha} e^{(r-\underline{\alpha})t}$. We thus have

$$\begin{aligned} U^H(x[B^H, B^L, \underline{\alpha}, \overline{\alpha}]) &\geq U_{\delta^H}(B, \underline{\alpha}) \\ &= \frac{B^{1-\gamma}}{1-\gamma} \frac{\underline{\alpha}^{1-\gamma}}{\underline{\alpha} + \delta^H - \underline{\delta}}, \quad \gamma \neq 1; \quad (161) \\ &= \frac{\delta^H \ln\left(B \underline{\delta}\right) + r - \underline{\delta}}{\delta^{H2}}, \quad \gamma = 1. \end{aligned}$$

Solving for the w^* such that (160)=(161) at $w = w^*$ yields

$$\begin{aligned} w^*(b) &= 1 - \frac{b^\gamma - b}{1-b} \frac{1}{1-\gamma} \frac{\underline{\alpha}}{\underline{\alpha} + \delta^H - \underline{\delta}}, \quad \gamma \neq 1; \quad (162) \\ &= 1 + \frac{b}{1-b} \frac{\ln(b)}{\delta^H}, \quad \gamma = 1. \end{aligned}$$

For the $\gamma \neq 1$ case, by (18) and (9), $\underline{\alpha} \in [\alpha^L, \alpha^H]$ implies that $\underline{\alpha} + \delta^H - \underline{\delta} > 0$. Also, since $b \in (0,1)$, $(b^\gamma - b)/(1-\gamma) > 0$: the numerator and denominator are both positive if $\gamma < 1$ and both negative if $\gamma > 1$. This establishes that $w^*(b) < 1$

for $b \in (0,1)$. Observe also that $w^*(\cdot)$ is continuous in b, with $w^*(0) = 1$ and, by L'Hôpital's Rule,

$$\lim_{b\to 1^-} \frac{b^\gamma - b}{1-b} = 1-\gamma \quad \Longrightarrow \quad \lim_{b\to 1^-} w^*(b) = 1 - \frac{\underline{\alpha}}{\underline{\alpha} + \delta^H - \underline{\delta}} < 1.$$

For the $\gamma = 1$ case, since $b \in (0,1)$, $\ln(b) < 0$. Also, $\delta^H > 0$ by (9). So $w^*(b) < 1$. Observe again that $w^*(\cdot)$ is continuous in b, with $\lim_{b\to 0^+} w^*(b) = 1$ and $\lim_{b\to 1^-} w^*(b) = 1 - \delta^H < 1$ by L'Hôpital's Rule.

Since increases in H's budget continuously strictly increase $\underline{x}^H$ early in time without affecting $\underline{x}^L$, $U^H\big(x[B^H(1-w), B^L, \underline{\alpha}, \infty]\big)$ continuously strictly decreases in w. Also,

$$U^H\big(x[B^H, B^L, \underline{\alpha}, \infty]\big) \geq U^H\big(x[B^H, B^L, \underline{\alpha}, \overline{\alpha}]\big),$$

since x^L is the same in each case, and

$$U^H\big(x[0, B^L, \underline{\alpha}, \infty]\big) < U^H\big(x[B^H, B^L, \underline{\alpha}, \overline{\alpha}]\big),$$

for the same reason. So, recalling that we can assume $B = 1$ without loss of generality due to homotheticity, there is a unique $w^H(b) \in (0,1)$ such that

$$U^H\big(x[(1-b)(1-w^H(b)), b, \underline{\alpha}, \infty]\big) = U^H\big(x[(1-b), b, \underline{\alpha}, \overline{\alpha}]\big).$$

Then by (160)–(162),

$$U^H\big(x[(1-b)(1-w^*(b)), b, \underline{\alpha}, \infty]\big) < U^H\big(x[(1-b), b, \underline{\alpha}, \overline{\alpha}]\big).$$

So $w^H(b) < w^*(b) < 1$ for all $b \in (0,1)$. Since $w^*(b)$ is continuous and $\lim_{b\to 1^-} w^*(b) < 1$, it only remains to show that $\lim_{b\to 0^+} w^H(b) < 1$.

To do this, observe that

$$\begin{aligned} U^H\big(x[B^H(1-w), B^L, \underline{\alpha}, \infty]\big) &\leq U_{\delta^H}\left(B^H(1-w) + B^L, \alpha^H\right) \\ &= \frac{\left(B^H(1-w) + B^L\right)^{1-\gamma}}{1-\gamma}\left(\alpha^H\right)^{-\gamma}, \qquad \gamma \neq 1; \qquad (163) \\ &= \frac{\delta^H \ln\left((B^H(1-w)+B^L)\delta^H\right) + r - \delta^H}{\delta^{H2}}, \quad \gamma = 1, \end{aligned}$$

with the first weak inequality following from the fact that the right-hand side is the maximum utility achievable for H with collective budget $B^H(1-w) + B^L$. Solving for the $\tilde{w}$ such that (163)=(161) at $w = \tilde{w}$ yields

$$\tilde{w}(b) = \frac{1}{1-b}\,\frac{\alpha^H\eta - \underline{\alpha}}{\alpha^H\eta}, \qquad (164)$$

where η equals (8) with $\alpha^H, \delta^H, \overline{\alpha}, \overline{\delta}$ in place of $\alpha, \delta, \tilde{\alpha}, \tilde{\delta}$.

As above, we must have $w^H(b) < \tilde{w}(b) < 1$ for all $b \in (0,1)$. Since $\tilde{w}(b)$ is continuous and $\tilde{w}(0) < 1$, $\lim_{b\to 0^+} w^H(b) < 1$, as desired. So $\sup_b w^H(b, \underline{\alpha}, \overline{\alpha}) < 1$ given $\underline{\alpha} \geq \alpha^L$ and $\underline{\alpha} \leq \alpha^H$.

Bounded WTP for H given spending maximum only

Given $B = 1$, H's payoff in the absence of constraints, after foregoing fraction w of her budget, can be found analytically:

$$\begin{aligned} U^H\big(x^*[(1-b)(1-w), b]\big) &= \int_0^\infty e^{-\delta^H t} u\big(x_t^*[(1-b)(1-w), b]\big) dt \\ &= \frac{b^{1-\gamma}}{1-\gamma}\big(\alpha^L\big)^{-\gamma}\Big(1 + \frac{(1-b)(1-w)\alpha^H}{b\alpha^L}\eta\Big)^{-\gamma}\Big(\frac{1-b}{b}\eta^\gamma + \frac{\alpha^L}{\alpha^L + \delta^H - \delta^L}\Big), \quad \gamma \neq 1; \\ &= \frac{1}{\delta^H}\Big(\ln\Big(b\delta^L\Big(1 + \frac{(1-b)(1-w)\alpha^H}{b\alpha^L}\eta\Big)\Big) + \frac{r - \delta^L}{\delta^H}\Big), \qquad \gamma = 1, \end{aligned} \tag{165}$$

where η equals (8) with α^H, δ^H, α^L, δ^L in place of α, δ, $\tilde{\alpha}$, $\tilde{\delta}$.

H's payoff given spending maximum $\overline{\alpha} \leq \alpha^H$ only is weakly greater than the payoff she receives when the collective budget $B = 1$ is spent at rate α^L (with equality only when b is sufficiently large: see (22)):

$$U^H(x[1-b, b, 0, \overline{\alpha}]) \geq U_{\delta^H}(1, \alpha^L). \tag{166}$$

Solving for the w^* such that (165)$=U_{\delta^H}(1, \alpha^L)$ at $w = w^*$ yields

$$w^*(b) = 1 - \frac{\alpha^L}{\alpha^H \eta} < 1, \tag{167}$$

independent of b.

$U^H\big(x^*[(1-b)(1-w), b]\big)$ strictly and continuously decreases in w, since $\partial U^H(x^*[B^H, B^L])/\partial B^H > 0$ everywhere (see (118)).

$$U^H(x^*[1-b, b]) > U^H\big(x[1-b, b, 0, \overline{\alpha}]\big),$$

by Proposition 4 and the fact that $\overline{x}^L$ is a best response to $\overline{x}^H$ even in the absence of constraints (see Appendix A.7), so that there would be alternative equilibrium of the Stackelberg game (with $x^H = \overline{x}^H$) if the inequality above failed. Finally,

$$U^H(x^*[0, b]) < U^H\big(x[1-b, b, 0, \overline{\alpha}]\big),$$

since $x^*[0, b] = x[0, b, 0, \overline{\alpha}]$ is the δ^L-optimal schedule given collective budget b, and $x_t[B^H, b, 0, \overline{\alpha}]$ strictly increases in B^H for all t. So there is a unique $w^H(b) \in (0, 1)$ such that

$$U^H(x^*[(1-b)(1-w^H(b)), b]) = U^H\big(x[1-b, b, 0, \overline{\alpha}]\big).$$

Then by (166)–(167),

$$U^H(x^*[(1-b)(1-w^*(b)), b]) \leq U^H\big(x[1-b, b, 0, \overline{\alpha}]\big).$$

So $w^H(b) \leq w^*(b) < 1$ for all $b \in (0, 1)$, and $w^H(b)$ is bounded below 1 as b approaches 0 or 1. So $\sup_b w^H(b, 0, \overline{\alpha}) < 1$ given $\overline{\alpha} \leq \alpha^H$.

Unbounded WTP for L given both constraints

Assume that $b < 1 - \alpha^L/\overline{\alpha}$, so that $\overline{t}^* \equiv \overline{t}^*[1-b, b(1-w)] > 0$ for all $w \in [0,1)$ (see (21) and the discussion around (22)).

Given $B = 1$ and a spending maximum $\overline{\alpha} \le \alpha^H$ only, L's payoff after foregoing fraction w of his budget can be found analytically:

$$U^L\big(x[(1-b), b(1-w), 0, \overline{\alpha}]\big) = \int_0^{\overline{t}^*} e^{-\delta^L t} u\big((1-b)\overline{\alpha} e^{(r-\overline{\alpha})t}\big) dt$$
$$+ \int_{\overline{t}^*}^{\infty} e^{-\delta^L t} u\Big(\big((1-b)e^{(r-\overline{\alpha})\overline{t}^*} + b(1-w)e^{r\overline{t}^*}\big) e^{(r-\alpha^L)(t-\overline{t}^*)}\Big) dt$$

$$= \frac{\big((1-b)\overline{\alpha}\big)^{1-\gamma}}{1-\gamma} \Big(\frac{1}{\overline{\alpha} + \delta^L - \overline{\delta}} + \Big(\frac{1}{\alpha^L} - \frac{1}{\overline{\alpha} + \delta^L - \overline{\delta}}\Big) \Big(\frac{1-b}{b(1-w)} \, \frac{\overline{\alpha} - \alpha^L}{\alpha^L}\Big)^{-\frac{\overline{\alpha} + \delta^L - \overline{\delta}}{\overline{\alpha}}} \Big), \quad \gamma \neq 1; \tag{168}$$

$$= \frac{1}{\delta^L} \Big(\ln\big((1-b)\overline{\delta}\big) + \frac{r - \overline{\delta}}{\delta^L} + \Big(\frac{1-b}{b(1-w)}\Big)^{-\frac{\delta^L}{\overline{\delta}}} \Big(\frac{\overline{\delta} - \delta^L}{\delta^L}\Big)^{\frac{\overline{\delta} - \delta^L}{\overline{\delta}}} \Big), \qquad \gamma = 1.$$

Observe that, by (9) and (18), $\overline{\alpha} + \delta^L - \overline{\delta} > 0$.

Let $\tilde{t}(b) \equiv -\ln(b) z$, where

$$z \equiv \begin{cases} \frac{\overline{\delta} - \delta^L}{2\overline{\alpha}(\overline{\delta} - \underline{\alpha} - \delta^L)}, & \underline{\alpha} < \overline{\delta} - \delta^L; \\ 1, & \underline{\alpha} = \overline{\delta} - \delta^L; \\ \infty, & \underline{\alpha} > \overline{\delta} - \delta^L. \end{cases} \tag{169}$$

L's payoff given both spending minimum $\underline{\alpha} \ge \alpha^L$ and spending maximum $\overline{\alpha} \in (\alpha^L, \alpha^H]$ can be upper-bounded by the sum of the following three terms. First, as a baseline, is the payoff he receives from H's schedule $\underline{\overline{x}}^H$ in isolation:

$$U_{\delta^L}(1-b, \overline{\alpha}) = \frac{(1-b)^{1-\gamma}}{1-\gamma} \frac{\overline{\alpha}^{1-\gamma}}{\overline{\alpha} + \delta^L - \overline{\delta}}, \qquad \gamma \neq 1; \tag{170}$$
$$= \frac{\delta^L \ln\big((1-b)\overline{\delta} + r - \overline{\delta}\big)}{\delta^{L2}}, \quad \gamma = 1.$$

Second is the additional payoff he would receive from his own spending from $t = 0$ to $\tilde{t}(b)$ if this spending confronted no diminishing marginal utility, so that each unit spent at t offered $u'(\underline{x}_t^H)$ units of flow utility:

$$\int_0^{\tilde{t}(b)} e^{-\delta^L t} b\underline{\alpha} e^{(r-\underline{\alpha})t}\big((1-b)\overline{\alpha}e^{(r-\overline{\alpha})t}\big)^{-\gamma} dt$$

$$= b\underline{\alpha}\big((1-b)\overline{\alpha}\big)^{-\gamma}\frac{1}{\underline{\alpha}+\delta^L-\overline{\delta}}\Big(1-b^{-(\overline{\delta}-\underline{\alpha}-\delta^L)z}\Big). \tag{171}$$

Third is the additional payoff he would receive from the present value of his budget not spent before $\tilde{t}(b)$, i.e. $be^{-\underline{\alpha}\tilde{t}(b)}$, if it were feasible for him to spend it subject to no constraints and subject to no diminishing marginal utility from L's own other spending. This can be found by subtracting $U_{\delta^L}(1-b,\overline{\alpha})$ from (168) and substituting $e^{-\underline{\alpha}\tilde{t}(b)}$ for $1-w$ in the remainder to get, for all γ,

$$\frac{\overline{\alpha}^{1-\gamma}}{\overline{\alpha}+\delta^L-\overline{\delta}}\Big(\frac{\overline{\alpha}-\alpha^L}{\alpha^L}\Big)^{\frac{\overline{\delta}-\delta^L}{\overline{\alpha}}}(1-b)^{\frac{\overline{\delta}-\delta^L-\gamma\overline{\alpha}}{\overline{\alpha}}}b^{(1+\underline{\alpha}z)\frac{\overline{\alpha}+\delta^L-\overline{\delta}}{\overline{\alpha}}}. \tag{172}$$

Denote the sum of (170)–(172) by $\tilde{U}^L(b)$, so that

$$U^L\big(x[1-b,b,\underline{\alpha},\overline{\alpha}]\big)\leq\tilde{U}^L(b), \tag{173}$$

and let $w^*(b)$ denote the value of w such that (168)=$\tilde{U}^L(b)$ at $w=w^*(b)$:

$$w^*(b) = 1-\Bigg((\overline{\alpha}+\delta^L-\overline{\delta})\Big(\frac{\alpha^L}{\overline{\alpha}-\alpha^L}\Big)^{\frac{\overline{\delta}-\delta^L}{\overline{\alpha}}}(1-b)^{\frac{\overline{\alpha}+\delta^L-\overline{\delta}}{\overline{\alpha}}} \tag{174}$$
$$\Big(\frac{\underline{\alpha}\big((1-b)\overline{\alpha}\big)^{-\gamma}}{\underline{\alpha}+\delta^L-\overline{\delta}}\big(b^{\frac{\overline{\delta}-\delta^L}{\overline{\alpha}}}-b^{\frac{\overline{\delta}-\delta^L}{\overline{\alpha}}-(\overline{\delta}-\underline{\alpha}-\delta^L)z}\big)$$
$$+\frac{\overline{\alpha}^{1-\gamma}}{\overline{\alpha}+\delta^L-\overline{\delta}}\Big(\frac{\overline{\alpha}-\alpha^L}{\alpha^L}\Big)^{\frac{\overline{\delta}-\delta^L}{\overline{\alpha}}}(1-b)^{\frac{\overline{\delta}-\delta^L-\gamma\overline{\alpha}}{\overline{\alpha}}}b^{(1+\underline{\alpha}z)\frac{\overline{\alpha}+\delta^L-\overline{\delta}}{\overline{\alpha}}}\Big)\Bigg)^{\frac{\overline{\alpha}}{\overline{\alpha}+\delta^L-\overline{\delta}}}$$

$\lim_{b\to 0} w^*(b)=1$ under all conditions.

Recall that $w^L(b,\underline{\alpha},\overline{\alpha})$ is the value of w such that

$$U^L\big(x[1-b,b(1-w),0,\overline{\alpha}]\big)=U^L\big(x[1-b,b,\underline{\alpha},\overline{\alpha}]\big).$$

$w^L(b,\underline{\alpha},\overline{\alpha})\in(0,1)$ is defined by the facts that

$$U^L\big(x[1-b,b,0,\overline{\alpha}]\big)>U^L\big(x[1-b,b,\underline{\alpha},\overline{\alpha}]\big)$$

(since L can implement $x[1-b,b,\underline{\alpha},\overline{\alpha}]$ in the presence of a spending maximum but no minimum, in the open-loop setting, but prefers not to (see Appendix A.7)); that

$$U^L\big(x[1-b,0,0,\overline{\alpha}]\big)<U^L\big(x[1-b,b,\underline{\alpha},\overline{\alpha}]\big);$$

and that $U^L\big(x[1-b,b(1-w),0,\overline{\alpha}]\big)$ strictly and continuously decreases in w.

Furthermore by (173) $w^L(b,\underline{\alpha},\overline{\alpha})\geq w^*(b)$ for all b. So $\lim_{b\to 0} w^L(b,\underline{\alpha},\overline{\alpha})=1$.

Unbounded WTP for L given spending minimum only

Given $B = 1$, L's payoff in the absence of constraints, after foregoing fraction w of his budget, can be found analytically:

$$U^L\big(x^*[1-b, b(1-w)]\big) = \int_0^\infty e^{-\delta^L t} u\big(x_t^*[1-b, b(1-w)]\big)dt$$

$$= \frac{1}{1-\gamma}\Bigg[\frac{\Big((1-b)\alpha^H + b(1-w)\alpha^L/\eta\Big)^{1-\gamma}}{\alpha^H+\delta^L-\delta^H}\Bigg(1-\Big(b(1-w)+\frac{(1-b)\alpha^H}{\alpha^L}\eta\Big)^{-\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H}}$$

$$\cdot\ \big(b(1-w)\big)^{\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H}}\Bigg) + (\alpha^L)^{-\gamma}\Big(b(1-w)+\frac{(1-b)\alpha^H}{\alpha^L}\eta\Big)^{-\gamma\frac{\alpha^L}{\alpha^H}}\big(b(1-w)\big)^{\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H}}\Bigg], \quad \gamma \neq 1;$$

$$= \frac{1}{\delta^L}\Bigg[\ln\Big((1-b)\delta^H + \frac{b(1-w)\delta^L}{\eta}\Big) + \frac{r-\delta^H}{\delta^L} \tag{175}$$

$$+ \frac{(\delta^H-\delta^L)^2}{\delta^H\delta^L}\Big(b(1-w)+\frac{(1-b)\delta^H}{\delta^L}\eta\Big)^{-\frac{\delta^L}{\delta^H}}\big(b(1-w)\big)^{\frac{\delta^L}{\delta^H}}\Bigg], \quad \gamma = 1,$$

where η equals (8) with α^H, δ^H, α^L, δ^L in place of α, δ, $\tilde{\alpha}$, $\tilde{\delta}$.

In the presence of a spending minimum $\underline{\alpha}$ only, the collective spending rate at time 0—$\underline{X}_0$—is weakly less than α^H, the spending rate at time 0 that H implements if the entire collective budget is hers. (If it were greater, then because $\underline{X}$ grows at rate $r-\alpha^H$ until $\underline{t}^* \leq \infty$ and then grows at $r-\underline{\alpha} \geq r-\alpha^H$, $\underline{X}$ would be unaffordable.) Thus we also have $\tilde{t}[b] \geq \underline{t}^*[b]$, where

$$\tilde{t}[b] \equiv \ln\Big(\frac{\alpha^H}{b\underline{\alpha}}\Big)/(\alpha^H-\underline{\alpha})$$

denotes the regime-change time t at which $\alpha^H e^{(r-\alpha^H)t} = b\underline{\alpha}e^{(r-\underline{\alpha})t}$.

So given spending minimum $\underline{\alpha}$ only, L's payoff is weakly less than $\tilde{U}^L(b)$, where $\tilde{U}^L(b)$ denotes the payoff he receives if $X_t = \alpha^H e^{(r-\alpha^H)t}$ for $t \in [0, \tilde{t}[b])$ and $= b\underline{\alpha}e^{(r-\underline{\alpha})t}$ for $t \in [\tilde{t}[b], \infty)$.

We will assume in the expression below that $\underline{\alpha} < \alpha^H$. We can ignore the $\underline{\alpha} = \alpha^H$ case because, for any $b > 0$,

$$U^L\big(x[1-b, b, \alpha^H, \infty]\big) < U^L\big(x[1-b, b, \underline{\alpha}, \infty]\big) \quad \forall \underline{\alpha} < \alpha^H :$$

this follows from the fact that in the constrained open-loop setting with $\underline{\alpha} < \alpha^H$ it is feasible for L, given $x^H = \underline{x}^H$, to implement the δ^H-optimal allocation of the collective budget, but he prefers not to do so (see Appendix A.8). We can thus upper-bound $U^L\big(x[1-b,b,\alpha^H,\infty]\big)$ by the expression below for any choice of $\underline{\alpha} \in (\alpha^L,\alpha^H)$.

$$U^L\big(x[1-b,b,\underline{\alpha},\infty]\big) \le \tilde{U}^L(b) \tag{176}$$

$$= \frac{1}{1-\gamma}\Big(\frac{\big(\alpha^H\big)^{1-\gamma}}{\alpha^H+\delta^L-\delta^H}\Big(1-\Big(\frac{b\underline{\alpha}}{\alpha^H}\Big)^{\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H-\underline{\alpha}}}\Big)+\frac{\big(\alpha^H\big)^{1-\gamma}}{\underline{\alpha}+\delta^L-\underline{\delta}}\Big(\frac{b\underline{\alpha}}{\alpha^H}\Big)^{\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H-\underline{\alpha}}}\Big), \quad \gamma \ne 1; \tag{177}$$

$$= \frac{1}{\delta^L}\Big(\ln\big(\delta^H\big)+\frac{\delta^H-\underline{\delta}}{\delta^L}\Big(\frac{b\underline{\delta}}{\delta^H}\Big)^{\frac{\delta^L}{\delta^H-\underline{\delta}}}+\frac{r-\delta^H}{\delta^L}\Big), \quad \gamma = 1.$$

Let $w^*(b)$ denote the value of $w \in (-\infty,1)$ such that (175)=(177) at $w=w^*(b)$. $w^*(b)$ is defined, since $U^L(x^*[1-b,b(1-w)])$ strictly and continuously decreases in w from the supremum feasible payoff at $w=-\infty$ (∞ if $\gamma \le 1$, 0 if $\gamma>1$) to $U_{\delta^L}(1-b,\delta^H)$ at $w=1$, and (177) is greater than $U_{\delta^L}(1-b,\alpha^H)$. Furthermore, there cannot be a sequence $b_n \to 0$ with $\lim_{n\to\infty} b_n(1-w^*(b_n)) = \underline{b} > 0$, because by the continuity of $U^L(x^*[B^H,B^L])$ in both budgets and the continuity of $\tilde{U}^L(b)$, this would imply

$$U^L\big(x^*[1,\underline{b}]\big) = \tilde{U}^L(0),$$

but $\tilde{U}^L(0) = U^L(x^*[1,0]) = U_{\delta^L}(1,\alpha^H)$, and $U^L(x^*[B^H,B^L])$ strictly increases in B^L (see Appendix A.5: Final period: Unique equilibrium schedule given placement of first announcement). So $\lim_{n\to\infty} b_n(1-w^*(b_n)) = 0$.

To find $w^*(b)$, set (175) = (177) and subtract $U_{\delta^L}(1,\alpha^H)$ from both sides. If $\underline{\alpha} \le \delta^H-\delta^L$, divide both sides by $b^{\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H-\underline{\alpha}}}$ and simplify to get

$$\frac{1}{\alpha^H+\delta^L-\delta^H}\bigg(\Big((1-b)\alpha^H+b\big(1-w^*(b)\big)\frac{\alpha^L}{\eta}\Big)^{1-\gamma}-\alpha^H\bigg)b^{-\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H-\underline{\alpha}}}$$

$$+\ \big(\alpha^L\big)^{-\gamma}\Big(1-\frac{\alpha^L}{\alpha^L+\delta^H-\delta^L}\,\frac{\alpha^H}{\alpha^H+\delta^L-\delta^H}\Big)b^{-\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H}\frac{\underline{\alpha}}{\alpha^H-\underline{\alpha}}}\big(1-w^*(b)\big)^{\frac{\alpha^H+\delta^L-\delta^H}{\alpha^H}}$$

$$= \Big(\frac{1}{\underline{\alpha}+\delta^L-\underline{\delta}}-\frac{1}{\alpha^H+\delta^L-\delta^H}\Big)(\alpha^H)^{-\frac{\underline{\alpha}+\delta^L-\underline{\delta}}{\alpha^H-\underline{\alpha}}}, \qquad \gamma \ne 1;$$

$$\bigg(\ln\Big((1-b)\delta^H+\frac{b\big(1-w^*(b)\big)\delta^L}{\eta}\Big)-\ln(\delta^H)\bigg)b^{-\frac{\delta^L}{\delta^H-\underline{\delta}}}$$

$$+\ \frac{(\delta^H-\delta^L)^2}{\delta^H\delta^L}\Big(b\big(1-w^*(b)\big)+\frac{(1-b)\delta^H}{\delta^L}\eta\Big)^{-\frac{\delta^L}{\delta^H}}b^{-\frac{\delta^L}{\delta^H}\frac{\underline{\delta}}{\delta^H-\underline{\delta}}}\big(1-w^*(b)\big)^{\frac{\delta^L}{\delta^H}}$$

$$= (\delta^H-\underline{\delta})\Big(\frac{\underline{\delta}}{\delta^H}\Big)^{\frac{\delta^L}{\delta^H-\underline{\delta}}}, \qquad \gamma = 1.$$

In each case, given $\underline{\alpha} \le \delta^H-\delta^L$, the exponent on b in the first term is greater than -1. So the limit of the first term as $b\to 0$ is finite, regardless of $w^*(\cdot)$, by L'Hôpital's

Rule. The coefficient to the left of the b in the second term is nonzero by (18). So, from the second term, unless $\lim_{b\to 0} w^*(b) = 1$, the limit of left-hand side as $b \to 0$ is not a finite constant, as the right-hand side is.

If $\underline{\alpha} > \delta^H - \delta^L$, instead divide both sides by b—i.e. multiply the expression above by $b^{\frac{\alpha^{\overline{H}}+\delta^L-\delta^H}{\alpha^H-\underline{\alpha}}-1}$. Note that if $\underline{\alpha} > \delta^H - \delta^L$, the exponent on b on the right-hand side is now positive, so the limit of the right-hand side as $b \to 0$ is zero. Here the limit of the first term on the left-hand side as $b \to 0$ is finite, regardless of $w^*(\cdot)$, by L'Hôpital's Rule, and limit of the second term is defined and finite only if $\lim_{b\to 0} w^*(b) = 1$.

So $\lim_{b\to 0} w^*(b) = 1$.

Recall that $w^L(b, \underline{\alpha}, \infty)$ is the value of w such that

$$U^L\big(x^*[1-b, b(1-w)]\big) = U^L\big(x[1-b, b, \underline{\alpha}, \infty]\big).$$

$w^L(b)$ is defined and less than 1 by the reasoning following (177), noting that $U^L\big(x[1-b, b, \underline{\alpha}, \infty]\big)$ too is greater than $U_{\delta^L}(1-b, \alpha^H)$. Then by (176) and the fact that $U^L(x^*[1-b, b(1-w)])$ decreases in w, $w^L(b, \underline{\alpha}, \infty) \geq w^*(b)$ for all b.

So $\lim_{b\to 0} w^L(b, \underline{\alpha}, \infty) = 1$.

A.11 Proof of Proposition 11

By (135) and the surrounding discussion, a collective schedule is efficient iff it is a measure-zero deviation from a collective schedule X that exhausts the collective budget and satisfies

$$X_t = X_0 e^{\frac{r}{\gamma}t}\big(ae^{-\delta^H t} + (1-a)e^{-\delta^L t}\big)^\gamma$$

for some X_0 and some $a \in [0,1]$ representing the weight placed on U^H (with weight $1-a$ placed on U^L). The result follows from comparing this expression to the expressions for $x[\underline{\alpha}, \overline{\alpha}, B^H, B^L]$ from Propositions 5 and 7–9.

A.12 Proof of Proposition 12

Suppose $\{u(\cdot)\}$ satisfies single crossing from X, and let $\underline{s} < \overline{s}$.

$$\frac{\partial U^H(X)}{\partial X_{\overline{s}}} \geq \frac{R_{\underline{s}}}{R_{\overline{s}}} \frac{\partial U^H(X)}{\partial X_{\underline{s}}} \tag{178}$$

$$\begin{aligned} \implies \exists \tilde{t} : \sum_{t=0}^{\tilde{t}-1} e^{-\sum_{q=1}^t \delta_q^H} \Big(\frac{\partial u_t(X)}{\partial X_{\overline{s}}} - \frac{R_{\overline{s}}}{R_{\underline{s}}} \frac{\partial u_t(X)}{\partial X_{\underline{s}}}\Big) \\ + \sum_{t=\tilde{t}}^{\infty} e^{-\sum_{q=1}^t \delta_q^H} \Big(\frac{\partial u_t(X)}{\partial X_{\overline{s}}} - \frac{R_{\overline{s}}}{R_{\underline{s}}} \frac{\partial u_t(X)}{\partial X_{\underline{s}}}\Big) \geq 0 \end{aligned} \tag{179}$$

with each term in the first sum non-positive and each term in the second sum non-negative, by (25), and at least one term in the second sum positive, by (26).

If (179) holds for $\tilde{t} = 0$, then we immediately have

$$\sum_{t=\tilde{t}}^{\infty} e^{-\sum_{q=1}^{t}\delta_q^L}\Big(\frac{\partial u_t(X)}{\partial X_{\overline{s}}} - \frac{R_{\overline{s}}}{R_{\underline{s}}}\frac{\partial u_t(X)}{\partial X_{\underline{s}}}\Big) > 0$$
$$\implies \frac{\partial U^L(X)}{\partial X_{\overline{s}}} > \frac{R_{\underline{s}}}{R_{\overline{s}}}\frac{\partial U^L(X)}{\partial X_{\underline{s}}}. \tag{180}$$

Otherwise, since $e^{\sum_{q=1}^{t}(\delta_q^H-\delta_q^L)} - 1$ equals 0 at $t = 0$ and strictly increases in t, (179) implies

$$\sum_{t=0}^{\tilde{t}-1}\Big(e^{-\sum_{q=1}^{t}\delta_q^L} - e^{-\sum_{q=1}^{t}\delta_q^H}\Big)\Big(\frac{\partial u_t(X)}{\partial X_{\overline{s}}} - \frac{R_{\overline{s}}}{R_{\underline{s}}}\frac{\partial u_t(X)}{\partial X_{\underline{s}}}\Big)$$
$$+\sum_{t=\tilde{t}}^{\infty}\Big(e^{-\sum_{q=1}^{t}\delta_q^L} - e^{-\sum_{q=1}^{t}\delta_q^H}\Big)\Big(\frac{\partial u_t(X)}{\partial X_{\overline{s}}} - \frac{R_{\overline{s}}}{R_{\underline{s}}}\frac{\partial u_t(X)}{\partial X_{\underline{s}}}\Big) \geq \tag{181}$$
$$\sum_{t=0}^{\tilde{t}-1}\Big(e^{\sum_{q=1}^{\tilde{t}}(\delta_q^H-\delta_q^L)} - 1\Big)e^{-\sum_{q=1}^{t}\delta_q^H}\Big(\frac{\partial u_t(X)}{\partial X_{\overline{s}}} - \frac{R_{\overline{s}}}{R_{\underline{s}}}\frac{\partial u_t(X)}{\partial X_{\underline{s}}}\Big)$$
$$+\sum_{t=\tilde{t}}^{\infty}\Big(e^{\sum_{q=1}^{\tilde{t}}(\delta_q^H-\delta_q^L)} - 1\Big)e^{-\sum_{q=1}^{t}\delta_q^H}\Big(\frac{\partial u_t(X)}{\partial X_{\overline{s}}} - \frac{R_{\overline{s}}}{R_{\underline{s}}}\frac{\partial u_t(X)}{\partial X_{\underline{s}}}\Big) \geq 0, \tag{182}$$

with inequality (181) holding strictly unless the difference in derivatives equals zero for all $t \neq \tilde{t}$, in which case inequality (182) holds strictly by (26).

So the left-hand side of (181) is positive. In conjunction with (179), this implies (180). This proves part (a).

In the generalized open-loop game, suppose the players choose a strategy profile x such that $x_{\overline{s}}^H > 0, x_{\underline{s}}^L > 0$ for $\overline{s} > \underline{s}$. Then either (178) holds, or H can increase her payoff by shifting her resources marginally from $\overline{s}$ to $\underline{s}$. If (178) holds, then by part (a), (180) holds, so L can increase his payoff by shifting his resources marginally from $\underline{s}$ to $\overline{s}$. In either case, x is not an equilibrium. This proves part (b).