Population ethics with thresholds

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Abstract. We propose a new class of social quasi-orderings in a variable-population setting. In order to declare one utility distribution at least as good as another, the critical-level utilitarian value of the former must reach or surpass the value of the latter. For each possible absolute value of the difference between the population sizes of two distributions to be compared, we specify a non-negative threshold level and a threshold inequality. This inequality indicates whether the corresponding threshold level must be reached or surpassed in the requisite comparison. All of these threshold critical-level utilitarian quasi-orderings perform same-number comparisons by means of the utilitarian criterion. In addition to this entire class of quasi-orderings, we axiomatize two important subclasses. The members of the first subclass are associated with proportional threshold functions, and the well-known critical-band utilitarian quasi-orderings are included in this subclass. The quasi-orderings in the second subclass employ constant threshold functions; the members of this second class have, to the best of our knowledge, not been examined so far. Furthermore, we characterize the members of our class that (i) avoid the repugnant conclusion; (ii) avoid the sadistic conclusions; and (iii) respect the mere-addition principle.
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1 Introduction

As global population dynamics shift dramatically, the welfare implications of changes in population size and individual well-being have become increasingly important. Public policies, ranging from health care and education to environmental regulations, can profoundly influence not only the quality of life for current generations but also who will exist in the future. Welfarist population ethics addresses these pressing issues, seeking to assess utility distributions across varying population sizes and compositions. The field's insights are vital for addressing real-world challenges, from resource allocation and reproductive rights to climate-change policies that will shape the lives of future generations. By providing frameworks to evaluate the moral weight of decisions that affect population size and individual welfare, population ethics can offer essential guidance for policymakers and ethicists alike.

A fundamental challenge in population ethics is how to extend welfare comparisons from fixed-population scenarios to those involving different population sizes. For instance, how can we evaluate a policy that leads to the emergence of one million additional individuals? This question exemplifies the challenges faced in population ethics. Common theories include average utilitarianism and classical (or total) utilitarianism, each with distinct implications for such scenarios. Alternative variable-population extensions of (generalized) utilitarian principles have been discussed by numerous scholars, including Hurka (1983), Blackorby and Donaldson (1984), Ng (1986), Blackorby, Bossert, and Donaldson (2005), Asheim and Zuber (2014, 2022), Pivato (2020), and Spears and Stefánsson (2021). These approaches can lead to markedly different policy recommendations, highlighting the importance of rigorous analysis in welfare economics for population changes.

Most existing contributions in population ethics impose completeness on a social goodness relation, requiring all distributions of well-being of all possible populations to be comparable. However, this assumption is not without its pitfalls. In a variable-population setting, individuals existing in one population may have no counterparts in another, making direct comparisons challenging; see, for example, Golosov, Jones, and Tertilt (2007) and Pérez-Nievas, Conde-Ruiz, and Giménez (2019). These differences in the group of existing individuals provide a compelling reason to reconsider the appeal of completeness.

We propose a novel approach to population ethics by developing a general class of quasi-orderings, thereby examining the implications of allowing some degree of non-comparability across different populations. The core of our proposal involves using a threshold for making welfare comparisons between populations of varying sizes. More precisely, if the difference between the social value of a utility distribution for one population and that of another meets or exceeds a threshold, we consider the former to be at least as good as the latter. This threshold-based approach allows for more nuanced comparisons while acknowledging the inherent difficulties in comparing distributions with different population sizes.

A similar approach using a threshold has been developed in the literature on individual decision-making (Luce, 1956; Scott and Suppes, 1958; Aleskerov, Bouyssou, and Monjardet, 2007; Salant, 2011; Barberà, de Clippel, Neme, and Rozen, 2022). As shown by Tyson (2008) and de Clippel and Rozen (2024), this well-established method is useful in order to incorporate the cognitive limitations of individuals. That is, if the difference between two options to be compared is very small, an individual might not be able to meaningfully distinguish them. Thus, it is reasonable to have non-comparability in such cases. In particular, Aleskerov, Bouyssou, and Monjardet (2007) and Frick (2016) examine a threshold function that depends on pairs of options under consideration or within specific menus. In spite of the similarities to threshold representation in individual decision-making, there are distinctive aspects of our threshold approach for welfare comparison, including population changes. We assume that the threshold level may depend on the population-size difference associated with two distributions to be compared. In the above-mentioned literature on individual decision making, there is no such distinction—the same threshold is applied universally. Our focus is on the role of incompleteness in variable-population comparisons—the restrictions of our goodness relations to same-number environments are assumed to be complete.

There are important merits in incorporating the possibility of incompleteness in population ethics. As Parfit (1976, 1982, 1984) points out, there are significant trade-offs that are difficult to overcome. One possible path towards a resolution consists of abandoning the completeness assumption imposed on social goodness relations. This direction is indeed alluded to by Parfit (1984, pp. 430–432) and pursued by other scholars, including Blackorby, Bossert, and Donaldson (1996), Broome (2004, 2009), Qizilbash (2007), Rabinowicz (2009), Gustafsson (2020), and Hájek and Rabinowicz (2022).

To illustrate the trade-offs, let us start by considering total utilitarianism. An important benchmark in population ethics is the notion of neutrality. A life is neutral if it is, from the viewpoint of the person leading it, neither better nor worse than a life without any experiences. As is common practice in population ethics, we use a utility level of zero to represent a neutral life. In this context, total utilitarianism has a distinctive feature: it considers the addition of an individual with a utility level above neutrality to any given population as desirable. This characteristic sets total utilitarianism apart from most other approaches.

The repugnant conclusion (Parfit, 1984; see also Parfit, 1976, 1982) is implied if, for any population size n, any arbitrarily high level of lifetime well-being ξ, and any level of utility ε above neutrality but arbitrarily close to it, there exists a larger population of size $m > n$ such that a utility distribution of size n in which everyone experiences a lifetime utility of ξ is considered worse than a population of size m in which everyone's lifetime well-being is equal to ε. In other words, the repugnant conclusion means that population size can always be substituted for quality of life, no matter how close to neutrality everyone's level of well-being may be.

Parfit (1984) shows that total utilitarianism implies the repugnant conclusion, which constitutes a serious shortcoming. Critical-level utilitarianism, proposed by Blackorby and Donaldson (1984), uses a fixed critical level to perform different-number comparisons. A critical level is a fixed number with the property that adding one person at this level leads to a utility distribution that is neither better nor worse than the original. Because we allow for non-comparability, adding someone at this level may lead to a distribution that is either as good as the original distribution, or the augmented distribution and the original are non-comparable. If completeness is assumed and the critical level is equal to zero (neutrality), total utilitarianism results. If the critical level exceeds the neutral level, the repugnant conclusion is avoided; see Blackorby and Donaldson (1984). According to critical-level utilitarianism, utility distributions are compared in terms of their respective sums of the differences between each individual utility level and the critical level.

Although critical-level utilitarian orderings with positive critical levels have numerous attractive features, the critical-level utilitarian class is not without its critics; see, for example, Arrhenius (2000) and Williamson (2021). However, there appears to be no definitive population principle that can reasonably be declared to be superior. Arguably, this lack of consensus cannot but emerge because the complexity of variable-population considerations leads to numerous impossibility results generated by the inherent difficulty of simultaneously satisfying seemingly plausible properties associated with population change. See, for instance, Carlson (1998) for a discussion of some trilemmas that emerge in population ethics.

Average utilitarianism constitutes an alternative to the critical-level approach. While per-capita evaluations (such those that employ the per-capita GDP of a country) continue to be commonly used in economic assessments, there is by now a broad consensus that comparisons based on averages often lead to profoundly counter-intuitive consequences that are difficult to justify. To illustrate the problems associated with averaging methods, observe that they do not have a fixed critical level. By definition of average utilitarianism, the addition of a person with a level of well-being slightly below the average utility of an existing population is considered detrimental, no matter how high this average may be. Likewise, the addition of a person with a level of well-being slightly above the average utility of an existing population is considered desirable, no matter how low this average may be. Clearly, these two observations—especially the latter—constitute unacceptable features of a population principle. Therefore, averaging methods are almost universally rejected in population ethics.

Two desiderata that are frequently consulted in population ethics are avoidance of the sadistic conclusion (Arrhenius, 2000) and the mere-addition principle (Parfit, 1984). The sadistic conclusion is implied if the addition of people with negative levels of well-being (below neutrality) to a given distribution can be considered better than the addition of (not necessarily the same number of) people with positive utility levels (above neutrality). Mere addition requires that adding people whose utilities are above neutrality to a given distribution cannot lead to a distribution that is worse than the original. In addition to the property of avoidance of the repugnant conclusion, we examine the axiom of avoidance of the sadistic conclusion and the mere-addition principle in the context of our new class of threshold critical-level utilitarian quasi-orderings.

Ours is not the first contribution that incorporates non-comparability in population ethics. The class of critical-band utilitarian criteria constitutes, as far as we are aware, the only incomplete approach to population ethics that has been examined axiomatically. A member of this class is identified by a non-degenerate and bounded interval such that a utility distribution is better than another if and only if the former is at least as good as the latter for all critical-level utilitarian orderings associated with the members of the interval. Relations generated by this principle are quasi-orderings (that is, reflexive and transitive) but not complete. A characterization of these quasi-orderings is provided by

Blackorby, Bossert, and Donaldson (1996, 2005).

Our general class of social quasi-orderings is broader than the class of critical-band utilitarian criteria. All members of our class employ a fixed critical level but, in contrast to the well-known critical-level utilitarian criteria, some utility distributions that involve different population sizes are not ranked. In order to declare one utility distribution at least as good as another, the critical-level utilitarian value of the former must reach or surpass the value of the latter. For each possible absolute value of the difference between the population sizes of two distributions to be compared, we specify a non-negative threshold level and an inequality that indicates whether this level must be reached or surpassed. All of these threshold critical-level utilitarian quasi-orderings perform same-number comparisons by means of the utilitarian criterion. In addition to the entire threshold critical-level utilitarian class, we axiomatize two important subclasses. The members of the first subclass are associated with proportional threshold functions, and the well-known critical-band utilitarian quasi-orderings are included in this subclass. The quasi-orderings in the second subclass employ constant threshold functions; the members of this second class have, to the best of our knowledge, not been examined so far.

Section 2 presents our basic definitions. In Section 3, we introduce the axioms that are employed in our theorems. The main result, a characterization of the class of threshold critical-level utilitarian quasi-orderings, follows in Section 4. A corollary provides an alternative axiomatization. Subclasses of the proportional threshold critical-level utilitarian quasi-orderings and the constant threshold critical-level utilitarian quasi-orderings are axiomatized in Section 5. The repugnant conclusion, the sadistic conclusion, and the mere-addition principle are examined in the context of threshold critical-level utilitarianism in Section 6. Section 7 concludes. The independence of the axioms employed in our main result is established in the Appendix.

2 Social quasi-orderings

The set of all positive (all non-negative) integers is denoted by $\mathbb{N}$ ($\mathbb{N}_0$). The set of all (all positive, all non-negative, all negative) real numbers is $\mathbb{R}$ ($\mathbb{R}_{++}$, $\mathbb{R}_+$, $\mathbb{R}_{--}$). For $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, we use $\mathbf{1}_n$ to denote the n-dimensional vector composed of n ones, and $\mathbf{1}_n^i$ is the i^{th} n-dimensional unit vector. The set of all utility distributions is $\Omega = \cup_{n \in \mathbb{N}} \mathbb{R}^n$. For $u \in \Omega$ and $t \in \mathbb{N} \setminus \{1\}$, u^t denotes a t-fold replica of u.

Our objective is to establish a social quasi-ordering (a reflexive and transitive binary relation) R on Ω. We interpret R as a goodness relation—that is, for any two distributions $u, v \in \Omega$, uRv means that u is considered at least as good as v. The asymmetric and symmetric parts P and I of R are defined as

$$\begin{aligned} uPv &\Leftrightarrow uRv \text{ and } \neg vRu; \\ uIv &\Leftrightarrow uRv \text{ and } vRu. \end{aligned}$$

As usual, P and I represent the betterness relation and the equal-goodness relation associated with the goodness relation R. Furthermore, we define the non-comparability relation N of a goodness relation R by

$$uNv \;\Leftrightarrow\; \neg uRv \text{ and } \neg vRu.$$

Clearly, if R is an ordering (a complete quasi-ordering), the corresponding relation N is empty.

Let $h\colon \mathbb{N}_0 \to \mathbb{R}_+$ be a threshold function that assigns a non-negative threshold level to every possible absolute value of the population-size difference associated with two distributions. To determine whether a distribution $u \in \Omega$ is to be considered at least as good as a distribution $v \in \Omega$, we need to specify whether the threshold has to be surpassed by the difference in the values assigned to u and to v, or the requisite difference is merely required to be greater than or equal to the threshold. Because the choice between a strict inequality and a weak inequality may depend on the difference in the population sizes associated with u and with v, we use a sequence $\langle \rhd^k \rangle_{k\in\mathbb{N}_0}$ of threshold inequalities, where $\rhd^k \in \{\geq, >\}$ is the threshold inequality that applies to the population-size difference $k \in \mathbb{N}_0$.

The following definition identifies all pairs $\big(h; \langle \rhd^k \rangle_{k\in\mathbb{N}_0}\big)$ of a threshold function and a sequence of threshold inequalities that may be employed in our class of social quasi-orderings.

Definition 1. *A pair $\big(h; \langle \rhd^k \rangle_{k\in\mathbb{N}_0}\big)$ of a threshold function h and a sequence of threshold inequalities $\langle \rhd^k \rangle_{k\in\mathbb{N}_0}$ satisfies size-difference consistency if*

(i) $h(0) = 0$ and $\rhd^0 = \geq$;

(ii) $h(|n-m|) + h(|m-\ell|) \geq h(|n-\ell|)$ for all $n, m, \ell \in \mathbb{N}$;

(iii) for all pairwise distinct $n, m, \ell \in \mathbb{N}$, if $\rhd^{|n-m|} = \rhd^{|m-\ell|} = \geq$, then

(iii.a) $\rhd^{|n-\ell|} = \geq$

or

(iii.b) $h(|n-m|) + h(|m-\ell|) > h(|n-\ell|)$.

Property (i) of this definition ensures that any two distributions of the same population size can be compared. Properties (ii) and (iii) guarantee that our relations are indeed transitive. Also, property (ii) implies that the function h must be subadditive on $\mathbb{N}$. To see that this is the case, let $k, k' \in \mathbb{N}$. Choose $n, m, \ell \in \mathbb{N}$ such that $n - m = k$ and $m - \ell = k'$. It follows that $k + k' = n - m + m - \ell = n - \ell$, and (ii) requires that $h(k) + h(k') \geq h(k + k')$ for all $k, k' \in \mathbb{N}$—that is, h is subadditive on $\mathbb{N}$.

Our new class of social quasi-orderings is defined as follows.

Definition 2. *A social quasi-ordering R on Ω is a threshold critical-level utilitarian quasi-ordering if there exist a critical level $\alpha \in \mathbb{R}$ and a size-different consistent pair $\big(h; \langle \rhd^k \rangle_{k\in\mathbb{N}_0}\big)$ of a threshold function h and a sequence of threshold inequalities $\langle \rhd^k \rangle_{k\in\mathbb{N}_0}$ such that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,*

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \rhd^{|n-m|} h(|n-m|).$$

Two important subclasses of these social quasi-orderings are (i) the proportional threshold critical-level utilitarian quasi-orderings, obtained if there exists $\beta \in \mathbb{R}_+$ such that $h(k) = \beta k$ for all $k \in \mathbb{N}_0$, and (ii) the constant threshold critical-level utilitarian quasi-orderings that result if there exists $\delta \in \mathbb{R}_+$ such that $h(0) = 0$ and $h(k) = \delta$ for all $k \in \mathbb{N}$. An example of a threshold critical-level utilitarian quasi-ordering that is neither a proportional nor a constant threshold critical-level utilitarian quasi-ordering is obtained if $h(k) = \sqrt{k}$ for all $k \in \mathbb{N}_0$.

Two interesting special cases that belong to both the proportional and the constant threshold critical-level utilitarian subclasses are the critical-level utilitarian orderings (Blackorby and Donaldson, 1984), and the variants that are obtained by replacing equal goodness with non-comparability across different population sizes. Critical-level utilitarianism is obtained if we choose $\rhd^k = \geq$ and $h(k) = 0$ for all $k \in \mathbb{N}$. The variant that we refer to as non-complete critical-level utilitarianism results for $\rhd^k = >$ and $h(k) = 0$ for all $k \in \mathbb{N}$—that is, the weak threshold inequalities of critical-level utilitarianism are replaced with strict inequalities, thereby generating instances of non-comparability. More explicitly, according to critical-level utilitarianism, we obtain

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - \alpha] \geq \sum_{i=1}^{m}[v_i - \alpha]$$

for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$. Non-complete critical-level utilitarianism is defined by letting

$$\begin{aligned} uRv \;\;\Leftrightarrow\;\; & \left[n = m \;\text{ and }\; \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i\right] \;\text{ or} \\ & \left[n \neq m \;\text{ and }\; \sum_{i=1}^{n}[u_i - \alpha] > \sum_{i=1}^{m}[v_i - \alpha]\right] \end{aligned}$$

for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$.

The class of proportional threshold critical-level utilitarian quasi-orderings contains all of the critical-band utilitarian quasi-orderings characterized by Blackorby, Bossert, and Donaldson (1996, 2005). They are obtained for the sequence of threshold inequalities that assigns a weak inequality to each population-size difference. According to Blackorby, Bossert, and Donaldson's (2005) definition, R is a critical-band utilitarian quasi-ordering if there exists a non-degenerate and bounded interval $Q \subseteq \mathbb{R}$ such that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$\begin{aligned} uRv \;\;\Leftrightarrow\;\; & \left[n = m \;\text{ and }\; \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i\right] \;\text{ or} \\ & \left[n \neq m \;\text{ and }\; \sum_{i=1}^{n}[u_i - c] \geq \sum_{i=1}^{m}[v_i - c] \;\text{ for all } c \in Q\right]. \end{aligned}$$

At first sight, it may seem surprising that each of these quasi-orderings is identical to one of the proportional threshold critical-level quasi-orderings. In the definition of critical-band utilitarianism, the non-degenerate interval with given endpoints may be open, closed, or half open. Because of the symmetry inherent in a threshold principle, the critical-band utilitarian quasi-orderings that are generated by half-open intervals may appear not to be covered by any of the proportional threshold critical-level quasi-orderings. However, as shown in Bossert, Cato, and Kamaga (2025), there is no such discrepancy. For any two given endpoints $c, c' \in \mathbb{R}$ with $c < c'$, the four critical-band quasi-orderings that correspond to the four possible intervals (c, c'), $(c, c']$, $[c, c')$, and $[c, c']$ are identical. Therefore, in showing that every critical-band quasi-ordering generated by a non-degenerate and bounded interval Q is a proportional threshold critical-level utilitarian quasi-ordering, we can, without loss of generality, assume that Q is a closed interval.

Theorem 1. *Suppose that $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_{++}$. The critical-band utilitarian quasi-ordering R' associated with the critical band $Q = [\alpha - \beta, \alpha + \beta]$ is equal to the proportional threshold critical-level utilitarian quasi-ordering R associated with the critical level α, the threshold function defined by $h(k) = \beta k$ for all $k \in \mathbb{N}_0$, and the sequence of threshold inequalities $\langle \rhd^k \rangle_{k \in \mathbb{N}_0}$ defined by $\rhd^k = \geq$ for all $k \in \mathbb{N}_0$.*

Proof. Suppose that $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_{++}$. Furthermore, suppose that the critical-band utilitarian quasi-ordering R' is associated with the critical band $Q = [\alpha - \beta, \alpha + \beta]$, and the proportional threshold critical-level utilitarian quasi-ordering R is associated with the critical level α, the threshold function defined by $h(k) = \beta k$ for all $k \in \mathbb{N}_0$, and the sequence of threshold inequalities $\langle \rhd^k \rangle_{k \in \mathbb{N}_0}$ defined by $\rhd^k = \geq$ for all $k \in \mathbb{N}_0$. We prove that $R = R'$.

Clearly, the desired conclusion follows immediately if $n = m$. Now suppose that $n, m \in \mathbb{N}$ are distinct, $u \in \mathbb{R}^n$, and $v \in \mathbb{R}^m$.

Case 1. $n > m$.

Suppose that uRv. By definition,

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \geq \beta(n - m) \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - (\alpha + \beta)] \geq \sum_{i=1}^{m}[v_i - (\alpha + \beta)]. \quad (1)$$

Since $n > m$, it follows that, for all $c \in (-\infty, \alpha + \beta]$,

$$\sum_{i=1}^{n}[u_i - c] \geq \sum_{i=1}^{m}[v_i - c]$$

and, therefore, $uR'v$ follows.

Now suppose that $uR'v$. This implies

$$\sum_{i=1}^{n}[u_i - (\alpha + \beta)] \geq \sum_{i=1}^{m}[v_i - (\alpha + \beta)].$$

From (1), we obtain uRv.

Case 2. $n < m$.

Suppose first that uRv. By definition,

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \geq (m-n)\beta \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - (\alpha - \beta)] \geq \sum_{i=1}^{m}[v_i - (\alpha - \beta)]. \tag{2}$$

Since $n < m$, it follows that, for all $c \in [\alpha - \beta, \infty)$,

$$\sum_{i=1}^{n}[u_i - c] \geq \sum_{i=1}^{m}[v_i - c]$$

and hence $uR'v$.

Finally, suppose that $uR'v$. It follows that

$$\sum_{i=1}^{n}[u_i - (\alpha - \beta)] \geq \sum_{i=1}^{m}[v_i - (\alpha - \beta)]$$

and, using (2), we obtain uRv. ■

The notion of k-critical sets plays a crucial role in our results. For all $k \in \mathbb{N}$ and for all $u \in \Omega$, the k-critical set of u for a social quasi-ordering R is defined as

$$Q^k(u) = \{c \in \mathbb{R} \mid (u, c\mathbf{1}_k)Iu \text{ or } (u, c\mathbf{1}_k)Nu\}.$$

The set $Q^k(u)$ contains all utility levels c such that, if k individuals are added to the utility distribution u and each of them has the utility level c, then the augmented distribution is either as good as u or non-comparable to u. Blackorby, Bossert, and Donaldson (1996, 2005, Chapter 7) propose an analogous definition of Q^1 but not of any Q^k for values of k above one. Thus, their formulation is restricted to population augmentations by a single individual only. Because they rule out equal goodness across different population sizes, Blackorby, Bossert, and Donaldson's critical sets are formulated exclusively in terms of non-comparability.

To link our definition of the threshold critical-level utilitarian quasi-orderings to these k-critical sets, suppose that R is a member of our class. Let $u \in \Omega$, $k \in \mathbb{N}$, and $c \in \mathbb{R}$. By definition,

$$(u, c\mathbf{1}_k)Iu \;\Leftrightarrow\; \rhd^k = \geq \;\text{ and } h(k) = 0$$

and

$$(u, c\mathbf{1}_k)Nu \;\Leftrightarrow\; \neg\big(k[c-\alpha] \rhd^k h(k)\big) \text{ and } \neg\big(k[\alpha - c] \rhd^k h(k)\big).$$

Therefore, whenever $h(k) = 0$ for some $k \in \mathbb{N}$, the k-critical set of $u \in \Omega$ for the corresponding threshold critical-level utilitarian quasi-ordering is given by the singleton set $Q^k(u) = \{\alpha\}$. If $h(k) \in \mathbb{R}_{++}$, the k-critical set of u for the threshold critical-level utilitarian quasi-ordering under consideration is

$$Q^k(u) = \left(\alpha - \frac{h(k)}{k}, \alpha + \frac{h(k)}{k}\right) \text{ or } Q^k(u) = \left[\alpha - \frac{h(k)}{k}, \alpha + \frac{h(k)}{k}\right],$$

depending on whether $\rhd^k = \geq$ or $\rhd^k = >$.

The proportional threshold critical-level utilitarian quasi-orderings such that $\beta \in \mathbb{R}_{++}$ are associated with the k-critical sets $Q^k(u) = \{\alpha\}$ if $h(k) = 0$, and

$$Q^k(u) = (\alpha - \beta, \alpha + \beta) \quad \text{or} \quad Q^k(u) = [\alpha - \beta, \alpha + \beta]$$

if $h(k) \in \mathbb{R}_{++}$, depending on whether $\rhd^k = \geq$ or $\rhd^k = >$. In these cases, the k-critical set is independent of k.

Finally, the k-critical sets for the constant threshold critical-level utilitarian quasi-orderings with a positive parameter δ are $Q^k(u) = \{\alpha\}$ if $h(k) = 0$, and

$$Q^k(u) = \left(\alpha - \frac{\delta}{k}, \alpha + \frac{\delta}{k}\right) \quad \text{or} \quad Q^k(u) = \left[\alpha - \frac{\delta}{k}, \alpha + \frac{\delta}{k}\right],$$

if $h(k) \in \mathbb{R}_{++}$, depending on whether $\rhd^k = \geq$ or $\rhd^k = >$. Thus, in the case of a positive-valued constant threshold function, the k-critical set shrinks in inverse proportion as k increases.

Note that these k-critical sets depend on the population-size difference k but not on the utility distribution $u \in \Omega$ under consideration. This feature is part of our axiomatization, and it is shared with the critical-band utilitarian quasi-orderings of Blackorby, Bossert, and Donaldson (1996, 2005).

3 Axioms

We begin with two axioms that are concerned with the comparison of utility distributions of the same population size. Recall that, when comparing such distributions, any threshold critical-level utilitarian quasi-ordering applies utilitarianism. Consequently, the two axioms are those that characterize utilitarianism for same-number comparisons. There is an extensive literature on axiomatic characterizations of utilitarianism and related welfare criteria. Since our aim in this paper is not to provide a new characterization of same-number utilitarianism itself, we adopt a simple formulation that appears in Blackorby, Bossert, and Donaldson (2002, 2005).

Minimal increasingness requires that, for any two utility distributions where everyone receives the same utility level, the distribution with the higher total utility is considered socially better. This axiom is also called egalitarian dominance. We note that the property is substantially weaker than the Pareto principle and, thus, it is rather uncontroversial.

Minimal increasingness. For all $n \in \mathbb{N}$ and for all $a, b \in \mathbb{R}$, if $a > b$, then $a\mathbf{1}_n P b\mathbf{1}_n$.

The following axiom is the second key property to characterize utilitarianism. It states that all utility gains and losses are equally valuable, regardless of who receives them. Thus, the gain of a certain amount of utility by a very rich individual is considered just as good as the same utility gain for a poor individual. We do not intend to claim that this is uncontroversial but, as long as utilitarianism is used for same-number comparisons, this axiom cannot but be satisfied as a necessary condition. Ooghe, Schokkaert, and Van de gaer (2007) refer to essentially the same axiom as utilitarian neutrality.

Incremental equity. For all $n \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, for all $d \in \mathbb{R}$, and for all $i, j \in \{1, \ldots, n\}$,

$$\left(u + d\mathbf{1}_n^i\right) I \left(u + d\mathbf{1}_n^j\right).$$

These two axioms characterize same-number utilitarianism; see Blackorby, Bossert, and Donaldson (2002, Theorem 10; 2005, Theorem 4.9). We note that any alternative set of axioms that characterizes same-number utilitarianism could be employed instead.

The remaining axioms that we employ are concerned with the comparison of utility distributions of different population sizes. To begin with, we assume that, for each positive population-size difference k, there is at least one utility distribution u such that the corresponding k-critical set is non-empty.

Non-emptiness of k-critical sets. For all $k \in \mathbb{N}$, there exists $u \in \Omega$ such that $Q^k(u)$ is non-empty.

Next, we assume that, for each positive value of k, there is at least one utility distribution u such that the k-critical set of u is bounded.

Boundedness of k-critical sets. For all $k \in \mathbb{N}$, there exists $u \in \Omega$ such that $Q^k(u)$ is bounded.

The conjunction of non-emptiness of k-critical sets and boundedness of k-critical sets parallels Blackorby, Bossert, and Donaldson's (2005) regularity of critical sets. Because they rule out singleton critical sets, Blackorby, Bossert, and Donaldson assume that the critical set contain at least two elements; in contrast, we merely require non-emptiness. Moreover, our axiom does not exclude the case where completeness is satisfied, while Blackorby, Bossert, and Donaldson's does.

The next property is responsible for the symmetry inherent in all threshold criteria. It postulates the existence of a utility level α and a utility distribution u with respect to which the k-critical set $Q^k(u)$, if it is non-empty, takes a symmetric form independently of the number k of added individuals. The symmetric structure of k-critical sets means that whether a population augmentation leads to an equally good or non-comparable distribution depends on how far apart the utility level c of added individuals is from the utility level α. Therefore, the axiom asserts that there exist a utility level α and a utility distribution u such that, for any population augmentation with individuals having the same utility level c, the augmentation is evaluated on the basis of the deviation of the utility of added people from the utility level α. In this sense, the utility level α plays the role of a reference utility level for evaluating population augmentations.

Symmetric critical-level population principle. There exist $\alpha \in \mathbb{R}$ and $u \in \Omega$ such that, for all $k \in \mathbb{N}$ and for all $c, c' \in \mathbb{R}$, if $c \in Q^k(u)$ and $|c' - \alpha| = |c - \alpha|$, then $c' \in Q^k(u)$.

Our next axiom requires that, for all positive values of k, the k-critical sets of all utility distributions coincide. This is analogous to Blackorby, Bossert, and Donaldson's (2005) critical-set population principle.

k-critical-set independence. For all $k \in \mathbb{N}$ and for all $u, v \in \Omega$, $Q^k(u) = Q^k(v)$.

Clearly, k-critical-set independence implies the existence of a k-critical set $\bar{Q}^k$ such that

$$\bar{Q}^k = Q^k(u) \text{ for all } u \in \Omega.$$

Combined with k-critical-set independence, the axioms of non-emptiness of k-critical sets and boundedness of k-critical sets together imply that, for all $k \in \mathbb{N}$, $\bar{Q}^k$ is non-empty and bounded.

Observe that if the k-critical sets $\bar{Q}^k$ were unbounded, then any two distributions with population-size difference k would be incomparable. This would be in conflict with the following intuition given by Broome (2009, p. 412), who writes that

> "There may be limits: perhaps it is a bad thing to add a person whose life would be miserable, and perhaps a good thing to add a person whose life would be wonderful. But the intuition is that, at least for a range of levels of wellbeing, adding a person within that range has neutral value."

The symmetric critical-level population principle by itself does not imply that α belongs to the k-critical set $Q^k(u)$. However, combined with other axioms, it follows that $\alpha \in Q^k(u)$, and that α is the midpoint of $Q^k(u)$; see Theorem 2 in the following section.

The next axiom imposes a consistency requirement on k-critical sets. By definition, a utility value c may belong to $Q^k(u)$ either because the distribution $(u, c\mathbf{1}_k)$ is as good as u, or because the two distributions in question are non-comparable. If the sets $Q^k(u)$ and $Q^k(v)$ are identical and contain a single element, it may be the case that equal goodness applies for u and non-comparability for v. As shown later, this discrepancy is ruled out whenever a k-critical set contains at least two elements; however, an additional axiom is required to deal with the singleton case.

k-critical-set singleton consistency. For all $k \in \mathbb{N}$, for all $u, v \in \Omega$, and for all $c \in \mathbb{R}$, if $Q^k(u) = Q^k(v) = \{c\}$, then

$$(u, c\mathbf{1}_k) I u \;\Leftrightarrow\; (v, c\mathbf{1}_k) I v.$$

The axiom of existence independence (Blackorby, Bossert, and Donaldson, 2005, Chapter 5) is an appealing separability property that is employed in some characterizations of the critical-level generalized-utilitarian orderings; see, for instance, Blackorby, Bossert, and Donaldson (2005, Theorem 6.10). An earlier use of this axiom can be found in Krantz, Luce, Suppes, and Tversky (1971), who examine its role in representations satisfying an Archimedean property; for an application to superior and inferior goods, see also Arrhenius and Rabinowicz (2005). In our context, it can be used to provide an interesting alternative characterization of our threshold critical-level utilitarian quasi-orderings. A detailed discussion of the property can be found in Blackorby, Bossert, and Donaldson (2005, Chapter 5).

Existence independence. For all $u, v, w \in \Omega$,

$$(u, w)R(v, w) \;\Leftrightarrow\; uRv.$$

Replication invariance is a well-established variable-population requirement; see Zoli (2009) and Bossert, Cato, and Kamaga (2023b, Lemma 1). It reflects the linearity associated with the proportional subclass of our threshold-based quasi-orderings.

Replication invariance. For all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, for all $v \in \mathbb{R}^m$, and for all $t \in \mathbb{N} \setminus \{1\}$,

$$uRv \;\Leftrightarrow\; u^t R v^t.$$

Finally, the axiom of critical-set midpoint consistency plays an important role in identifying the subclass of our quasi-orderings with a constant threshold function. Consider a distribution u of population size n, and another distribution v of size $m < n$. The axiom requires that if u is augmented by adding an individual whose utility level is equal to the midpoint of the critical set $Q^{n-m}(u)$, the relative goodness of the original distributions is unaffected.

Critical-set midpoint consistency. For all $n, m \in \mathbb{N}$ such that $n > m$, for all $u \in \mathbb{R}^n$ such that $Q^{n-m}(u)$ is non-empty and bounded, and for all $v \in \mathbb{R}^m$,

$$uRv \;\Leftrightarrow\; (u, (\sup Q^{n-m}(u) + \inf Q^{n-m}(u))/2)Rv.$$

If this axiom is combined with existence of k-critical sets, boundedness of k-critical sets, k-critical-set independence, and the symmetric critical-level population principle, the midpoint of $Q^{n-m}(u) = \bar{Q}^{n-m}$ is given by the utility level α with respect to which any k-critical set is symmetric. Also, note that if adding an individual with utility α results in a distribution that is as good as the original, critical-set midpoint consistency is satisfied. Therefore, the axiom generalizes the defining property of Blackorby and Donaldson's (1984) critical level.

4 A general characterization result

This section contains the statement and proof of our main result. Moreover, we present a corollary that may provide further insights into the roles played by some of the axioms employed.

Theorem 2. *A social quasi-ordering R on Ω satisfies minimal increasingness, incremental equity, non-emptiness of k-critical sets, boundedness of k-critical sets, the symmetric critical-level population principle, k-critical-set independence, and k-critical-set singleton consistency if and only if R is a threshold critical-level utilitarian quasi-ordering.*

Proof. 'If.' Suppose that R is a threshold critical-level utilitarian quasi-ordering.

We first prove that R is transitive. Let $n, m, \ell \in \mathbb{N}$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, and $w \in \mathbb{R}^\ell$ be such that uRv and vRw. If $n = m$ or $m = \ell$ or $n = \ell$, uRw follows trivially. Suppose now that n, m, ℓ are pairwise distinct. By definition,

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \rhd^{|n-m|} h(|n-m|)$$

and

$$\sum_{i=1}^{m}[v_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] \rhd^{|m-\ell|} h(|m-\ell|).$$

Adding these two inequalities implies

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] \geq h(|n-m|) + h(|m-\ell|); \tag{3}$$

clearly, (3) is true for all possible choices of $\rhd^{|n-m|}$ and $\rhd^{|m-\ell|}$. By part (ii) of size-difference consistency, it follows that

$$h(|n-m|) + h(|m-\ell|) \geq h(|n-\ell).\tag{4}$$

If at least one of (3) and (4) is satisfied with a strict inequality, it follows that

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] > h(|n-\ell|)$$

and uRw follows, no matter whether $\rhd^{|n-\ell|} = \geq$ or $\rhd^{|n-\ell|} = >$ is true.

Finally, suppose that both (3) and (4) are satisfied with an equality. Observe that (3) can only be satisfied with an equality if both $\rhd^{|n-m|}$ and $\rhd^{|m-\ell|}$ are equal to $\geq$. Given that (4) is satisfied with an equality, part (iii) of size-difference consistency implies that $\rhd^{|n-\ell|} = \geq$ so that uRw follows.

That minimal increasingness and incremental equity are satisfied follows because the same-number restrictions of these quasi-orderings are utilitarian as a consequence of part (i) of size-difference consistency; see Blackorby, Bossert, and Donaldson (2005, Theorem 4.9).

For all $k \in \mathbb{N}$ and for all $u \in \Omega$, the k-critical set $Q^k(u)$ of $u \in \Omega$ is given by the singleton set $\{\alpha\}$ if $h(k) = 0$, or by one of the intervals $(\alpha - h(k)/k, \alpha + h(k)/k)$ and $[\alpha - h(k)/k, \alpha + h(k)/k]$ if $h(k) \in \mathbb{R}_{++}$. Thus, in all cases, $Q^k(u)$ is non-empty and bounded and, therefore, non-emptiness of k-critical sets and boundedness of k-critical sets are satisfied. Moreover, each possible $Q^k(u)$ is symmetric with midpoint α so that the symmetric critical-level population principle is satisfied. It is immediate that $Q^k(u) = Q^k(v)$ for all $k \in \mathbb{N}$ and for all $u, v \in \Omega$ and, therefore, k-critical-set independence is satisfied. Because the threshold functions and the threshold inequalities only depend on the difference in population sizes and not on the utility distributions, k-critical-set singleton independence is satisfied.

'Only if.' Suppose that R satisfies the axioms of the theorem statement.

Step 1. The conjunction of minimal increasingness and incremental equity implies that all same-number comparisons must employ the utilitarian criterion. Consistent with this observation, we define $\rhd^0 = \geq$ and $h(0) = 0$. The requisite characterization is established by Blackorby, Bossert, and Donaldson (2002, Theorem 10); see also Blackorby, Bossert, and Donaldson (2005, Theorem 4.9). Although the theorem of Blackorby, Bossert, and Donaldson is stated for social orderings, its proof does not require the assumption of completeness so that it can be applied to social quasi-orderings.

Step 2. Combined with k-critical-set independence, non-emptiness of k-critical sets and boundedness of k-critical sets together imply that, for all $k \in \mathbb{N}$, there exists a non-empty and bounded k-critical set $\bar{Q}^k \subseteq \mathbb{R}$ satisfying $\bar{Q}^k = Q^k(u)$ for all $u \in \Omega$. Suppose that $\alpha \in \mathbb{R}$ is defined as in the statement of the symmetric critical-level population principle. This implies that, for all $k \in \mathbb{N}$ and for all $c, c' \in \mathbb{R}$,

$$\left[c \in \bar{Q}^k \text{ and } |c' - \alpha| = |c - \alpha|\right] \Rightarrow c' \in \bar{Q}^k. \tag{5}$$

Furthermore, for all $k \in \mathbb{N}$, if $\bar{Q}^k$ has at least two elements, then it follows that, for all $c \in \bar{Q}^k$ and for all $u \in \Omega$,

$$(u, c\mathbf{1}_k)Nu. \tag{6}$$

To prove (6), let $c \in \bar{Q}^k$ and $u \in \Omega$. Because $c \in \bar{Q}^k = Q^k(u)$, it follows that $(u, c\mathbf{1}_k)Iu$ or $(u, c\mathbf{1}_k)Nu$. Suppose that $(u, c\mathbf{1}_k)Iu$. Since $\bar{Q}^k$ contains at least two elements, there exists $c' \in \bar{Q}^k$ with $c \neq c'$. Assume, without loss of generality, that $c < c'$. From **Step 1**, we obtain $(u, c'\mathbf{1}_k)P(u, c\mathbf{1}_k)$. By transitivity, $(u, c'\mathbf{1}_k)Pu$, which is a contradiction because $c \in \bar{Q}^k$. Thus, $(u, c\mathbf{1}_k)Nu$ must hold.

We now derive the structure of the critical sets $\bar{Q}^k$ and the threshold inequalities $\rhd^k$. Let $k \in \mathbb{N}$. If $\bar{Q}^k$ is a singleton, it follows that $\bar{Q}^k = \{\alpha\}$ by (5). By k-critical-set singleton consistency, we obtain either $(u, \alpha\mathbf{1}_k)Iu$ for all $u \in \Omega$ or $(u, \alpha\mathbf{1}_k)Nu$ for all $u \in \Omega$. If $(u, \alpha\mathbf{1}_k)Iu$ for all $u \in \Omega$, define $\rhd^k = \geq$; if $(u, \alpha\mathbf{1}_k)Nu$ for all $u \in \Omega$, let $\rhd^k = >$.

Now suppose that $\bar{Q}^k$ contains at least two distinct elements c and c'. Without loss of generality, suppose that $c < c'$, and let $c'' \in \mathbb{R}$ be such that $c < c'' < c'$. Let $u \in \Omega$. From (6), it follows that $(u, c\mathbf{1}_k)Nu$ and $(u, c'\mathbf{1}_k)Nu$. Using same-number utilitarianism and transitivity, we obtain $(u, c'\mathbf{1}_k)P(u, c''\mathbf{1}_k)P(u, c\mathbf{1}_k)$. If $(u, c''\mathbf{1}_k)Ru$, transitivity implies $(u, c'\mathbf{1}_k)Pu$, contradicting the hypothesis that $c' \in \bar{Q}^k$. Analogously, if $uP(u, c''\mathbf{1}_k)$, transitivity implies $uP(u, c\mathbf{1}_k)$, again a contradiction. Thus, $(u, c''\mathbf{1}_k)Nu$ and, therefore, $c'' \in \bar{Q}^k$ so that $\bar{Q}^k$ is an interval. Let $\alpha \in \mathbb{R}$ be as defined in the symmetric critical-level population principle so that it satisfies (5), and let $c \in \bar{Q}^k \setminus \{\alpha\}$. By (5), the number c' is an element of $\bar{Q}^k$ if $|c' - \alpha| = |c - \alpha|$; that is, if $\alpha = (c + c')/2$. Thus, $\bar{Q}^k$ is a symmetric interval with midpoint α. The symmetry of $\bar{Q}^k$ implies that the interval must be either open or closed and, therefore, there exists $\gamma_k \in \mathbb{R}_{++}$ such that

$$\bar{Q}^k = (\alpha - \gamma_k, \alpha + \gamma_k) \text{ or } \bar{Q}^k = [\alpha - \gamma_k, \alpha + \gamma_k].$$

We define $\rhd^k = \geq$ if $\bar{Q}^k$ is open, and $\rhd^k = >$ if $\bar{Q}^k$ is closed.

Step 3. Let $u \in \Omega$. We prove that (i) $(u, c\mathbf{1}_k)Pu$ for all $c \in \mathbb{R}$ such that $c > \hat{c}$ for all $\hat{c} \in \bar{Q}^k$, and that (ii) $uP(u, c\mathbf{1}_k)$ for all $c \in \mathbb{R}$ such that $c < \hat{c}$ for all $\hat{c} \in \bar{Q}^k$. To prove (i), let $c > \hat{c}$ for all $\hat{c} \in \bar{Q}^k$. Note that $c \notin \bar{Q}^k$. Because $\bar{Q}^k = Q^k(u)$ for all $u \in \Omega$, it follows that $(u, c\mathbf{1}_k)Pu$ or $uP(u, c\mathbf{1}_k)$. From **Step 1**, we obtain $(u, c\mathbf{1}_k)P(u, \hat{c}\mathbf{1}_k)$ for all $\hat{c} \in \bar{Q}^k$. If $uP(u, c\mathbf{1}_k)$, transitivity implies that $uP(u, \hat{c}\mathbf{1}_k)$ for all $\hat{c} \in \bar{Q}^k$. By definition of $\bar{Q}^k$, this is a contradiction. Thus, $(u, c\mathbf{1}_k)Pu$. The proof of (ii) is analogous.

Step 4.a. Let $k \in \mathbb{N}$. We show that if $\bar{Q}^k = \{\alpha\}$ and $\rhd^k = \geq$, then, for all $n, m \in \mathbb{N}$ such that $|n - m| = k$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uRv \Leftrightarrow \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \geq 0. \tag{7}$$

Suppose that $n > m$. To prove (7), observe first that the assumption $\rhd^k = \geq$ implies that $(v, \alpha\mathbf{1}_{n-m})Iv$. By transitivity, uIv if and only if $(v, \alpha\mathbf{1}_{n-m})Iu$ which, because all same-number comparisons are based on the utilitarian criterion, is equivalent to

$$\sum_{i=1}^{n} u_i = \sum_{i=1}^{m} v_i + (n-m)\alpha.$$

Thus, we obtain

$$uIv \Leftrightarrow \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] = 0. \tag{8}$$

Next, suppose that $\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] > 0$. Let $\varepsilon > 0$ be such that $\sum_{i=1}^{n} u_i = \sum_{i=1}^{m} v_i + (n-m)(\alpha + \varepsilon)$. Same-number utilitarianism implies $uI(v, (\alpha + \varepsilon)\mathbf{1}_{n-m})$, and the result of **Step 3** implies $(v, (\alpha + \varepsilon)\mathbf{1}_{n-m})Pv$. Transitivity implies uPv. Therefore, combined with (8), we obtain (7).

The proof of the claim for $n < m$ is analogous.

Step 4.b. Let $k \in \mathbb{N}$. Define γ_k as in **Step 2** and suppose that $\bar{Q}^k$ contains at least two elements and $\rhd^k = \geq$. Note that $\bar{Q}^k$ is open; see **Step 2**. We show that, for all $n, m \in \mathbb{N}$ such that $|n - m| = k$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uRv \Leftrightarrow \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \geq |n - m|\gamma_{|n-m|}. \tag{9}$$

To prove (9), we first show that, for all $n, m \in \mathbb{N}$ such that $|n-m| = k$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \geq |n-m|\gamma_{|n-m|} \;\Rightarrow\; uPv. \tag{10}$$

Suppose that $n > m$. Assume that

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \geq |n-m|\gamma_{|n-m|}.$$

This implies

$$\sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i + (n-m)(\alpha + \gamma_{|n-m|}).$$

Same-number utilitarianism implies $uR(v, (\alpha+\gamma_{|n-m|})\mathbf{1}_{n-m})$. Note that $(\alpha+\gamma_{|n-m|}) \notin \bar{Q}^k$. Thus, the result of **Step 3** implies that $(v, (\alpha + \gamma_{|n-m|}\mathbf{1}_{n-m}))Pv$. By transitivity, uPv follows. The proof of the claim for $n < m$ is analogous.

To complete the proof of (9), assume that $n > m$. Suppose, by way of contradiction, that uRv and

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] < |n-m|\gamma_{|n-m|}.$$

Then, there exists $\varepsilon \in \mathbb{R}_{++}$ such that

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] = |n-m|(\gamma_{|n-m|} - \varepsilon). \tag{11}$$

If $\varepsilon \geq 2\gamma_{|n-m|}$, then

$$\sum_{i=1}^{m}[v_i - \alpha] - \sum_{i=1}^{n}[u_i - \alpha] = |n-m|(\delta - \gamma_{|n-m|}) \geq |n-m|\gamma_{|n-m|},$$

and thus, vPu follows from (10). However, this is a contradiction because uRv. Thus, we assume $\varepsilon < 2\gamma_{|n-m|}$. From (11), we obtain

$$\sum_{i=1}^{n} u_i = \sum_{i=1}^{m} v_i + |n-m|(\alpha + \gamma_{|n-m|} - \varepsilon).$$

Same-number utilitarianism implies $uI(v, (\alpha+\gamma_{|n-m|}-\varepsilon)\mathbf{1}_{|n-m|})$. By transitivity, $(v, (\alpha+\gamma_{|n-m|} - \varepsilon)\mathbf{1}_{|n-m|})Rv$. From the result of **Step 2**, this is a contradiction because $(\alpha + \gamma_{|n-m|} - \varepsilon) \in \bar{Q}^k$. The proof of the claim for $n < m$ is analogous.

Step 4.c. Let $k \in \mathbb{N}$. Suppose that $\rhd^k = >$ and define γ_k as in **Step 2** if $\bar{Q}^k$ contains at least two elements, and let $\gamma_k = 0$ if $\bar{Q}^k = \{\alpha\}$. Note that $\bar{Q}^k$ is closed; see **Step 2**.

Thus, $(\alpha+\gamma_k) \in \bar{Q}^k$. We show that, for all $n, m \in \mathbb{N}$ such that $|n-m| = k$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] > |n-m|\gamma_{|n-m|}. \tag{12}$$

We first show that, for all $n, m \in \mathbb{N}$ such that $|n-m| = k$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] > |n-m|\gamma_{|n-m|} \;\Rightarrow\; uPv. \tag{13}$$

Suppose that $n > m$. Assume that

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] > |n-m|\gamma_{|n-m|}.$$

This implies

$$\sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i + |n-m|(\alpha + \gamma_{|n-m|} + \varepsilon)$$

for a sufficiently small $\varepsilon \in \mathbb{R}_{++}$. Note that $(\alpha + \gamma_{|n-m|} + \varepsilon) \notin \bar{Q}^k$. Thus, applying **Step 3** and same-number utilitarianism, the same argument as in **Step 4.b** establishes uPv. The proof of the claim for $n < m$ is analogous.

Finally, to complete the proof of (12), assume that $n > m$. Suppose, by way of contradiction, that uRv and

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \leq |n-m|\gamma_{|n-m|}. \tag{14}$$

Then, we obtain

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] = |n-m|(\gamma_{|n-m|} - \varepsilon)$$

for some $\varepsilon \in \mathbb{R}_{+}$. If $\varepsilon > 2\gamma_{|n-m|}$, we obtain vPu from (13). This is a contradiction because uRv. Furthermore, if $\varepsilon \leq 2\gamma_{|n-m|}$, the same argument as in **Step 4.b** provides a contradiction. The proof of the claim for $n < m$ is analogous.

Step 5. Define $h(k) = 0$ if $\bar{Q}^k$ is the singleton set $\{\alpha\}$, and $h(k) = k\gamma_k \in \mathbb{R}_{++}$ if $\bar{Q}^k$ contains more than one element. Combining our earlier observations, it follows that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \rhd^{|n-m|} h(|n-m|).$$

We complete the proof by showing that size-difference consistency is satisfied.

Part (i) of the property follows from same-number utilitarianism and the resulting definitions of $\rhd^0 = \geq$ and $h(0) = 0$.

Suppose that part (ii) of size-difference consistency is violated. Then there exist $n, m, \ell \in \mathbb{N}$ such that

$$h(|n-m|) + h(|m-\ell|) < h(|n-\ell|). \tag{15}$$

Let $n, m, \ell \in \mathbb{N}$, and let $v \in \mathbb{R}^m$ be arbitrary. Define $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^\ell$ such that

$$\sum_{i=1}^{n}[u_i - \alpha] = \sum_{i=1}^{m}[v_i - \alpha] + h(|n-m|) + \varepsilon \quad \text{and} \quad \sum_{i=1}^{\ell}[w_i - \alpha] = \sum_{i=1}^{m}[v_i - \alpha] - h(|m-\ell|) - \varepsilon,$$

where $\varepsilon \in \mathbb{R}_{++}$. It follows that

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] = h(|n-m|) + \varepsilon > h(|n-m|)$$

and

$$\sum_{i=1}^{m}[v_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] = h(|m-\ell|) + \varepsilon > h(|m-\ell|).$$

By definition, uRv and vRw, irrespective of the values of $\rhd^{|n-m|}$ and $\rhd^{|m-\ell|}$. Using the definitions of u and w in terms of v, it follows that

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] = h(|n-m|) + h(|m-\ell|) + 2\varepsilon.$$

By transitivity, we must have uRw, which implies

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] \geq h(|n-\ell|). \tag{16}$$

For sufficiently small ε, (15) implies

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] = h(|n-m|) + h(|m-\ell|) + 2\varepsilon < h(|n-\ell|),$$

contradicting (16). Therefore, part (ii) of size-difference consistency is satisfied.

Finally, we establish part (iii). Suppose that $n, m, \ell \in \mathbb{N}$ are pairwise distinct and that $\rhd^{|n-m|} = \rhd^{|m-\ell|} = \geq$.

Let $v \in \mathbb{R}^m$ be arbitrary, and define $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^\ell$ such that

$$\sum_{i=1}^{n}[u_i - \alpha] = \sum_{i=1}^{m}[v_i - \alpha] + h(|n-m|) \quad \text{and} \quad \sum_{i=1}^{\ell}[w_i - \alpha] = \sum_{i=1}^{m}[v_i - \alpha] - h(|m-\ell|).$$

It follows that

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] = h(|n-m|)$$

and

$$\sum_{i=1}^{m}[v_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] = h(|m-\ell|).$$

Because $\rhd^{|n-m|} = \rhd^{|m-\ell|} = \geq$, this implies uRv and vRw. By transitivity, we obtain uRw which, in turn, requires that

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] = h(|n-m|) + h(|m-\ell|) \rhd^{|n-\ell|} h(|n-\ell|).$$

This is impossible if $h(|n-m|) + h(|m-\ell|) = h(|n-\ell|)$ and $\rhd^{|n-\ell|} = >$ and, therefore, (iii) must be satisfied. ■

The independence of the axioms used in the above theorem is established in the Appendix.

The axiom of existence independence can be employed to provide an alternative axiomatization of our class. All threshold critical-level utilitarian quasi-orderings satisfy existence independence. Moreover, in the presence of minimal increasingness and incremental equity, existence independence implies k-critical-set independence and k-critical-set singleton consistency. The following lemma proves these claims.

Lemma 1. *(a) All threshold critical-level utilitarian quasi-orderings satisfy existence independence.*

(b) If a social quasi-ordering R on Ω satisfies minimal increasingness, incremental equity, and existence independence, then R satisfies k-critical-set independence and k-critical-set singleton consistency.

Proof. (a) Suppose that R is a threshold critical-level utilitarian quasi-ordering. Let $n, m, \ell \in \mathbb{N}$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, and $w \in \mathbb{R}^\ell$. By definition,

$$\begin{aligned} (u,w)R(v,w) \quad &\Leftrightarrow \quad \sum_{i=1}^{n}[u_i - \alpha] + \sum_{i=1}^{\ell}[w_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] - \sum_{i=1}^{\ell}[w_i - \alpha] \\ &\qquad \rhd^{|n+\ell-m-\ell|} h(|n+\ell-m-\ell|) \\ &\Leftrightarrow \quad \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \rhd^{|n-m|} h(|n-m|) \\ &\Leftrightarrow \quad uRv, \end{aligned}$$

as was to be proven.

(b) Suppose that R is a social quasi-ordering that satisfies minimal increasingness, incremental equity, and existence independence.

To prove that R satisfies k-critical-set independence, suppose that, by way of contradiction, there exist $n, m \in \mathbb{N}$, $u \in \mathbb{R}^n$, and $v \in \mathbb{R}^m$ such that $Q^{|n-m|}(u) \neq Q^{|n-m|}(v)$. This implies that there exists $c \in \mathbb{R}$ such that, without loss of generality, $c \in Q^{|n-m|}(u)$ and $c \notin Q^{|n-m|}(v)$. By existence independence,

$$(u, c\mathbf{1}_{|n-m|})Ru \;\Leftrightarrow\; (u, c\mathbf{1}_{|n-m|}, v)R(u, v). \tag{17}$$

By minimal increasingness and incremental equity, same-number comparisons must be made using utilitarianism and, therefore, $(u, c\mathbf{1}_{|n-m|}, v)I(u, v, c\mathbf{1}_{|n-m|})$. Combined with (17), it follows that

$$(u, c\mathbf{1}_{|n-m|})Ru \;\Leftrightarrow\; (u, v, c\mathbf{1}_{|n-m|})R(u, v).$$

Analogously, we obtain

$$(v, c\mathbf{1}_{|n-m|})Rv \;\Leftrightarrow\; (u, v, c\mathbf{1}_{|n-m|})R(u, v)$$

and, combining these two equivalences,

$$(u, c\mathbf{1}_{|n-m|})Ru \;\Leftrightarrow\; (v, c\mathbf{1}_{|n-m|})Rv. \tag{18}$$

Because $c \in Q^{|n-m|}(u)$ and $c \notin Q^{|n-m|}(v)$, it follows that

$$\left[(u, c\mathbf{1}_{|n-m|})Iu \text{ or } (u, c\mathbf{1}_{|n-m|})Nu\right] \quad \text{and} \quad \left[(v, c\mathbf{1}_{|n-m|})Pv \text{ or } vP(v, c\mathbf{1}_{|n-m|})\right],$$

contradicting (18).

Finally, we prove that R satisfies k-critical-set singleton consistency. Suppose that, by way of contradiction, there exist $n, m \in \mathbb{N}$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, and $c \in \mathbb{R}$ such that $Q^{|n-m|}(u) = Q^{|n-m|}(v) = \{c\}$ and, moreover,

$$(u, c\mathbf{1}_{|n-m|})Iu \;\text{ and }\; (v, c\mathbf{1}_{|n-m|})Nv.$$

This contradicts (18), which follows in analogy to the argument that establishes k-critical-set independence. ■

Combining Theorem 2 and Lemma 1, we obtain the following corollary.

Corollary 1. *A social quasi-ordering R on Ω satisfies minimal increasingness, incremental equity, non-emptiness of k-critical sets, boundedness of k-critical sets, the symmetric critical-level population principle, and existence independence if and only if R is a threshold critical-level utilitarian quasi-ordering.*

The axioms used in this corollary are independent. This observation is established by means of the quasi-orderings R^1 to R^6 that appear in the Appendix.

To conclude this section, we examine some fundamental characteristics of the critical sets that correspond to the threshold critical-level utilitarian quasi-orderings. Repeated application of part (ii) of size-difference consistency yields

$$h(rk) \leq h(k) + h((r-1)k) \leq h(k) + [h(k) + h((r-2)k)] \leq \cdots \leq rh(k),$$

so that $h(rk) \leq rh(k)$ for all $k, r \in \mathbb{N}$. This inequality is equivalent to

$$\frac{h(rk)}{rk} \leq \frac{h(k)}{k}.$$

For all $k \in \mathbb{N}$, the k-critical set $\bar{Q}^k$ of a threshold critical-level utilitarian quasi-ordering is given by

$$\bar{Q}^k = \left(\alpha - \frac{h(k)}{k}, \alpha + \frac{h(k)}{k}\right) \quad \text{or} \quad \bar{Q}^k = \left[\alpha - \frac{h(k)}{k}, \alpha + \frac{h(k)}{k}\right],$$

which implies that $cl(\bar{Q}^k) \supseteq cl(\bar{Q}^{rk})$ for all $k, r \in \mathbb{N}$, where cl denotes the closure operator on $\mathbb{R}$. Setting $k = 1$, it follows that

$$cl(\bar{Q}^1) \supseteq cl(\bar{Q}^r)$$

for all $r \in \mathbb{N}$. This means that $cl(\bar{Q}^1)$ is essentially the largest critical set and, therefore, $cl(\bar{Q}^k)$ cannot expand beyond $cl(\bar{Q}^1)$ as the size difference k increases.

However, this does not mean that the sequence $\langle cl(\bar{Q}^k)\rangle_{k\in\mathbb{N}}$ is nested in the sense that $cl(\bar{Q}^k) \supseteq cl(\bar{Q}^{k+1})$ for all $k \in \mathbb{N}$. As a counterexample, let h be the threshold function defined by $h(0) = 0$, $h(1) = 1$, $h(2) = 1$, $h(3) = 2$, and $h(k) = k/2$ for all $k \geq 4$. With a suitably chosen sequence of threshold inequalities $\langle \rhd^k \rangle_{k\in\mathbb{N}_0}$, the pair $\big(h; \langle \rhd^k \rangle_{k\in\mathbb{N}_0}\big)$ satisfies the three parts of size-difference consistency. The sequence composed of the radii of $cl(\bar{Q}^k)$ is

$$\left\langle \frac{h(k)}{k} \right\rangle_{k\in\mathbb{N}} = \left\langle 1, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \ldots \right\rangle$$

and, thus, the critical set $\bar{Q}^2$ is a strict subset of the critical set $\bar{Q}^3$.

A related result on subadditive functions appears in Fekete (1923). He shows that if a function $f\colon \mathbb{N} \to \mathbb{R}$ is subadditive, then the limit of the sequence

$$\left\langle \frac{f(k)}{k} \right\rangle_{k\in\mathbb{N}}$$

as k approaches infinity exists and is given by

$$\lim_{k\to\infty} \frac{f(k)}{k} = \inf_{k\in\mathbb{N}} \frac{f(k)}{k};$$

see also, for example, Romik (2014, p. 333) and Nathanson (2017, p. 247). The result applies in our setting and, therefore, we can conclude that, as the size difference k becomes large, the difference between the critical sets $\bar{Q}^k$ becomes small.

5 Proportional and constant thresholds

The two theorems of this section characterize subclasses of the proportional threshold critical-level utilitarian quasi-orderings and of the constant threshold critical-level utilitarian quasi-orderings. The reason why some members of the requisite subclass are excluded is that either of the additional axioms employed implies that the threshold inequalities must be the same for all positive-valued population-size differences.

A subclass of the proportional threshold critical-level utilitarian quasi-orderings is obtained if replication invariance is added to the axioms of Theorem 2, and a subclass of the constant threshold critical-level utilitarian quasi-orderings results if, instead of replication invariance, critical-set midpoint consistency is employed.

Theorem 3. *A social quasi-ordering R on Ω satisfies minimal increasingness, incremental equity, non-emptiness of k-critical sets, boundedness of k-critical sets, the symmetric critical-level population principle, k-critical-set independence, k-critical-set singleton consistency, and replication invariance if and only if R is a proportional threshold critical-level utilitarian quasi-ordering such that $\rhd^k = \geq$ for all $k \in \mathbb{N}$ or $\rhd^k = >$ for all $k \in \mathbb{N}$.*

Proof. 'If.' That all proportional threshold critical-level utilitarian quasi-orderings (and, thus, those for which $\rhd^k = \geq$ for all $k \in \mathbb{N}$ or $\rhd^k = >$ for all $k \in \mathbb{N}$) satisfy the axioms of the theorem statement other than replication invariance follows from Theorem 2.

To prove that replication invariance is satisfied, suppose that R is a proportional threshold critical-level utilitarian quasi-ordering such that $\rhd^k = \geq$ for all $k \in \mathbb{N}$ or $\rhd^k = >$ for all $k \in \mathbb{N}$. Let $n, m \in \mathbb{N}$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, and $t \in \mathbb{N} \setminus \{1\}$.

If $n = m$, we obtain

$$\begin{aligned} uRv \;&\Leftrightarrow\; \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{n}[v_i - \alpha] \rhd^0 0 \\ &\Leftrightarrow\; t\sum_{i=1}^{n}[u_i - \alpha] - t\sum_{i=1}^{n}[v_i - \alpha] \rhd^0 0 \\ &\Leftrightarrow\; u^t R v^t \end{aligned}$$

so that replication invariance is satisfied in this case.

If $n \neq m$, it follows that

$$\begin{aligned} uRv \;&\Leftrightarrow\; \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \rhd^{|n-m|} \beta|n-m| \\ &\Leftrightarrow\; t\sum_{i=1}^{n}[u_i - \alpha] - t\sum_{i=1}^{m}[v_i - \alpha] \rhd^{t|n-m|} t\beta|n-m| \\ &\Leftrightarrow\; u^t R v^t \end{aligned}$$

which is true because $\rhd^{|n-m|} = \rhd^{t|n-m|}$.

'Only if.' Suppose that R is a social quasi-ordering that satisfies the axioms. By Theorem 2, R is a threshold critical-level utilitarian quasi-ordering. Let $k \in \mathbb{N}$, and let $n, m \in \mathbb{N}$ be such that $n - m = k$. By definition,

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \rhd^k h(k)$$

for all $u \in \mathbb{R}^n$ and for all $v \in \mathbb{R}^m$. Let $t \in \mathbb{N} \setminus \{1\}$. Replication invariance requires that

$$\begin{aligned}\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \rhd^k h(k) \quad &\Leftrightarrow \quad t\sum_{i=1}^{n}[u_i - \alpha] - t\sum_{i=1}^{m}[v_i - \alpha] \rhd^{tk} h(tk) \\ &\Leftrightarrow \quad \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] \rhd^{tk} h(tk)/t \qquad (19)\end{aligned}$$

for all $u \in \mathbb{R}^n$ and for all $v \in \mathbb{R}^m$.

If $h(k) \neq h(tk)/t$ (without loss of generality, suppose that $h(k) < h(tk)/t$), we can choose u and v so that

$$h(k) < \sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] < h(tk)/t,$$

in contradiction to (19); note that this contradiction emerges for all possible combinations of the values of $\rhd^k$ and $\rhd^{tk}$. Therefore, $h(k) = h(tk)/t$, which implies that $h(tk) = th(k)$. This equality is true for all possible values of k and t and, setting $k = 1$ and defining $\beta = h(1) \in \mathbb{R}_+$, it follows that $h(t) = \beta t$ for all $t \in \mathbb{N} \setminus \{1\}$. Therefore, $h(k) = \beta k$ for all $k \in \mathbb{N}_0$.

Now suppose that, without loss of generality, $\rhd^k = \geq$ and $\rhd^{tk} = >$. Let u and v be such that

$$\sum_{i=1}^{n}[u_i - \alpha] - \sum_{i=1}^{m}[v_i - \alpha] = \beta k.$$

Substituting into (19), it follows that uRv and $\neg(u^t R v^t)$, a contradiction. Therefore, $\rhd^k = \rhd^{tk}$. Because k and t were chosen arbitrarily, we can set $k = 1$ to conclude that $\rhd^t = \rhd^1$ for all $t \in \mathbb{N} \setminus \{1\}$. Thus, all inequalities that apply to positive population-size differences must be the same. ■

Theorem 4. *A social quasi-ordering R on Ω satisfies minimal increasingness, incremental equity, non-emptiness of k-critical sets, boundedness of k-critical sets, the symmetric critical-level population principle, k-critical-set independence, k-critical-set singleton consistency, and critical-set midpoint consistency if and only if R is a constant threshold critical-level utilitarian quasi-ordering such that $\rhd^k = \geq$ for all $k \in \mathbb{N}$ or $\rhd^k = >$ for all $k \in \mathbb{N}$.*

Proof. 'If.' That the members of the class of constant threshold critical-level utilitarian quasi-orderings identified in the theorem statement satisfy all axioms other than critical-set midpoint consistency follows from Theorem 2.

To prove that critical-set midpoint consistency is satisfied, suppose that R is a constant threshold critical-level utilitarian quasi-ordering such that $\rhd^k = \geq$ for all $k \in \mathbb{N}$ or $\rhd^k = >$ for all $k \in \mathbb{N}$. For all $k \in \mathbb{N}$, let $\bar{Q}^k$ be the critical set for R. Assume that $n, m \in \mathbb{N}$ are such that $n > m$, and let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$. Because $(\sup \bar{Q}^{n-m} + \inf \bar{Q}^{n-m})/2 = \alpha$ and $\rhd^{n-m} = \rhd^{n-m+1}$, it follows that

$$\begin{aligned} uRv &\Leftrightarrow \sum_{i=1}^{n}[u_i-\alpha]-\sum_{i=1}^{m}[v_i-\alpha] \rhd^{n-m} \delta \\ &\Leftrightarrow \sum_{i=1}^{n}[u_i-\alpha]+[\alpha-\alpha]-\sum_{i=1}^{n}[v_i-\alpha] \rhd^{n-m+1} \delta \\ &\Leftrightarrow (u,(\sup \bar{Q}^{n-m}+\inf \bar{Q}^{n-m})/2)Rv \end{aligned}$$

so that midpoint consistency is satisfied.

'Only if.' Suppose that R is a social quasi-ordering that satisfies the axioms. By Theorem 2, R is a threshold critical-level utilitarian quasi-ordering. Let $k \in \mathbb{N}$. Clearly, the critical set $\bar{Q}^k$ for R is non-empty and bounded. Let $n, m \in \mathbb{N}$ be such that $n-m=k$.

By definition,

$$uRv \Leftrightarrow \sum_{i=1}^{n}[u_i-\alpha]-\sum_{i=1}^{m}[v_i-\alpha] \rhd^{k} h(k)$$

for all $u \in \mathbb{R}^n$ and for all $v \in \mathbb{R}^m$. Because $(\sup \bar{Q}^k+\inf \bar{Q}^k)/2=\alpha$, midpoint consistency requires that

$$\sum_{i=1}^{n}[u_i-\alpha]-\sum_{i=1}^{m}[v_i-\alpha] \rhd^{k} h(k) \Leftrightarrow \sum_{i=1}^{n}[u_i-\alpha]+[\alpha-\alpha]-\sum_{i=1}^{m}[v_i-\alpha] \rhd^{k+1} h(k+1). \quad (20)$$

If $h(k) \neq h(k+1)$ (without loss of generality, suppose that $h(k) < h(k+1)$), we can choose u and v so that

$$h(k) < \sum_{i=1}^{n}[u_i-\alpha]-\sum_{i=1}^{m}[v_i-\alpha] < h(k+1),$$

in contradiction to (20); as in the proof of Theorem 3, this contradiction emerges for all possible combinations of the values of $\rhd^k$ and $\rhd^{k+1}$. Therefore, $h(k)=h(k+1)$. Because $k \in \mathbb{N}$ was chosen arbitrarily, setting $\delta = h(1) \in \mathbb{R}_+$ implies that $h(k)=\delta$ for all $k \in \mathbb{N}$.

Now suppose that, without loss of generality, $\rhd^k = \geq$ and $\rhd^{k+1} = >$. Let u and v be such that

$$\sum_{i=1}^{n}[u_i-\alpha]-\sum_{i=1}^{m}[v_i-\alpha] = \delta.$$

Substituting into (20), it follows that uRv and $\neg((u,\alpha)Rv)$, a contradiction. Therefore, $\rhd^k = \rhd^{k+1}$. Because k was chosen arbitrarily, this implies that all inequalities that apply to positive population-size differences must be the same. ■

6 Properties of population quasi-orderings

There are numerous contributions that assess social orderings by means of properties that are deemed important in population ethics. A prominent example consists of Parfit's

(1984) observation that classical (or total) utilitarianism implies the repugnant conclusion. The repugnant conclusion is implied if, for any population size n, for any arbitrarily high level of well-being ξ, and for any level of utility ε above neutrality but arbitrarily close to it, there is a larger population size $m > n$ such that the distribution in which m people experience a level of well-being of ε is considered better than the distribution in which n people have a utility level of ξ. In other words, the repugnant conclusion means that population size can always be substituted for quality of life, no matter how close to neutrality everyone's level of well-being may be. The critical-level utilitarian orderings with a critical level above neutrality avoid the repugnant conclusion, as demonstrated by Blackorby and Donaldson (1984). The same is true for critical-level generalized utilitarianism, which employs transformed utilities in place of the individual utility levels. Critical-level generalized utilitarianism is obtained if the utility values u_i and v_i as well as the critical level α are replaced with $g(u_i)$, $g(v_i)$, and $g(\alpha)$ in the definition of critical-level utilitarianism, where $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous and increasing function such that $g(0) = 0$. (Generalized) classical utilitarianism results if the critical level α is equal to zero—the level of utility that represents a neutral life.

Alternative conditions for population-ethical assessments include avoidance of the sadistic conclusion (Arrhenius, 2000) and the mere-addition principle (Parfit, 1984). A social relation implies the sadistic conclusion if the addition of people with negative levels of well-being (below neutrality) can be considered better than the addition of people with positive utility levels. Mere addition demands that adding people with positive utility levels to a given population cannot lead to a distribution that is worse than the original. Of the critical-level (generalized) utilitarian orderings, only total (generalized) utilitarianism avoids the sadistic conclusion; if the critical level is above or below the level of neutrality, the sadistic conclusion is obtained. Mere addition is satisfied for non-positive critical levels.

It is well-recognized that it is difficult to avoid the repugnant conclusion and the sadistic conclusion and, at the same time, satisfy the mere-addition principle. For example, Ng (1989) proves that any ordering satisfying some quite uncontroversial properties violates the mere-addition principle or entails the repugnant conclusion; see also Carlson (1998). Blackorby, Bossert, and Donaldson (2005, Theorem 5.4) essentially show that no ordering that uses utilitarian same-number comparisons can avoid both the repugnant conclusion and the sadistic conclusion; the only additional assumption they employ is a minimal one-person trade-off condition that is implied by the existence of (not necessarily fixed) critical levels. Blackorby, Bossert, and Donaldson (2004), Shinotsuka (2008), and Bossert (2022) address this issue in the context of critical-level (generalized) utilitarianism; see also Arrhenius (2000), Bossert, Cato, and Kamaga (2023a), and Cato and Harada (2023) for related observations. It is possible to avoid both the repugnant conclusion and the sadistic conclusion by employing the maximin ordering, which violates strong Pareto (Bossert, 1990), or a discontinuous criterion (Bossert, Cato, and Kamaga, 2023b). As long as one commits to utilitarian same-number comparisons, impossibility results are difficult to avoid, provided that social relations are assumed to be complete.

Allowing for the existence of non-comparable utility distributions is indeed suggested by Parfit (1984, 2016), and his proposal uses a lexicographic criterion. To the best of our knowledge, the conjunction of the above-described three desiderata for variable-population social evaluation has not been examined thoroughly in the context of social quasi-orderings that employ same-number utilitarian comparisons. However, Blackorby, Bossert, and Donaldson (1997) do identify the members of the critical-band utilitarian class that avoid both the repugnant and the sadistic conclusion. This section shows how our threshold critical-level utilitarian quasi-orderings fare when assessed in terms of the three properties.

We begin with the repugnant conclusion. In line with the informal description provided above, the repugnant conclusion is implied by a quasi-ordering R if, for all $n \in \mathbb{N}$, for all $\xi \in \mathbb{R}_{++}$, and for all $\varepsilon \in (0, \xi)$, there exists $m \in \mathbb{N}$ with $m > n$ such that $\varepsilon \mathbf{1}_m P \xi \mathbf{1}_n$. Negating this statement leads to the axiom of avoidance of the repugnant conclusion.

Avoidance of the repugnant conclusion. There exist $n \in \mathbb{N}$, $\xi \in \mathbb{R}_{++}$, and $\varepsilon \in (0, \xi)$ such that

$$\neg \left(\varepsilon \mathbf{1}_m P \xi \mathbf{1}_n\right) \quad \text{for all } m \in \mathbb{N} \text{ with } m > n.$$

The subclass of threshold critical-level utilitarian quasi-orderings that avoid the repugnant conclusion are characterized in the following theorem.

Theorem 5. *A threshold critical-level utilitarian quasi-ordering R satisfies avoidance of the repugnant conclusion if and only if*

$$\alpha > 0$$

or

$$\alpha \le 0 \textit{ and } h(k) > -\alpha k \textit{ for all } k \in \mathbb{N}. \tag{21}$$

Proof. Suppose that R is a threshold critical-level utilitarian quasi-ordering. By definition, R avoids the repugnant conclusion if and only if there exist $n \in \mathbb{N}$, $\xi \in \mathbb{R}_{++}$, and $\varepsilon \in (0, \xi)$ such that

$$m(\varepsilon - \alpha) - n(\xi - \alpha) < h(m - n) \quad \text{for all } m \in \mathbb{N} \text{ with } m > n. \tag{22}$$

The condition expressed in (22) applies to the case in which the threshold inequality $\rhd^{m-n}$ is equal to $\geq$. This assumption involves no loss of generality because the same argument can be employed if the requisite inequality is strict.

Suppose first that $\alpha > 0$. Let $n \in \mathbb{N}$ be arbitrary, and choose $\xi \in \mathbb{R}_{++}$ and $\varepsilon \in (0, \xi)$ so that $\xi > \alpha > \varepsilon$. As a result of this choice, the left side of (22) is negative for all $m \in \mathbb{N}$ with $m > n$ and, because $h(m-n)$ (and, therefore, the right side of (22)) is positive, (22) is true, proving that the repugnant conclusion is avoided.

Now suppose that $\alpha \le 0$. We show that the existence of $n \in \mathbb{N}$, $\xi \in \mathbb{R}_{++}$, and $\varepsilon \in (0, \xi)$ such that (22) is satisfied is equivalent to the existence of $\bar{n} \in \mathbb{N}$ and $\bar{\xi} \in \mathbb{R}_{++}$ such that

$$(m - \bar{n})(\bar{\xi} - \alpha) < h(m - \bar{n}) \quad \text{for all } m \in \mathbb{N} \text{ with } m > \bar{n}. \tag{23}$$

Suppose first that there exist $\bar{n} \in \mathbb{N}$ and $\bar{\xi} \in \mathbb{R}_{++}$ such that (23) is satisfied. Let $n = \bar{n}$, and choose $\xi \in \mathbb{R}_{++}$ and $\varepsilon \in (0, \xi)$ such that

$$\varepsilon < \bar{\xi} < \xi.$$

These inequalities imply that

$$(m-n)(\bar{\xi}-\alpha) = m(\bar{\xi}-\alpha) - n(\bar{\xi}-\alpha) > m(\varepsilon-\alpha) - n(\xi-\alpha) \text{ for all } m \in \mathbb{N} \text{ with } m > n$$

and, combined with (23), it follows that

$$m(\varepsilon-\alpha) - n(\xi-\alpha) < h(m-n) \text{ for all } m \in \mathbb{N} \text{ with } m > n$$

so that (22) is satisfied.

Now suppose that there exist $n \in \mathbb{N}$, $\xi \in \mathbb{R}_{++}$, and $\varepsilon \in (0, \xi)$ such that (22) is satisfied. Define

$$\hat{m} = \min\{m > n \mid m(\varepsilon-\alpha) - n(\xi-\alpha) > 0\}.$$

Because the expression $m(\varepsilon-\alpha) - n(\xi-\alpha)$ is increasing and unbounded from above in m, $\hat{m}$ is well-defined.

If $\hat{m} > n+1$, define

$$h_{\min} = \min\{h(m-n) \mid m \in \{n+1, \ldots, \hat{m}-1\}\}.$$

Because the set $\{n+1, \ldots, \hat{m}-1\}$ is non-empty and finite, $h_{\min}$ is well-defined.

Now let $\bar{n} = n$, and define $\bar{\xi} \in \mathbb{R}_{++}$ such that

$$\bar{\xi} - \alpha < \frac{\hat{m}(\varepsilon-\alpha) - n(\xi-\alpha)}{\hat{m}-n} \tag{24}$$

and, if $\hat{m} > n+1$,

$$\bar{\xi} - \alpha < \frac{h_{\min}}{\hat{m}-1-n}. \tag{25}$$

Observe that $\bar{\xi} - \alpha > 0$ because $\alpha \leq 0$. From (24), it follows that

$$\bar{\xi} - \alpha < \frac{m(\varepsilon-\alpha) - n(\xi-\alpha)}{m-n} \text{ for all } m \in \mathbb{N} \text{ with } m \geq \hat{m} > n \tag{26}$$

because the fraction on the right side of (26) is increasing in m. Furthermore, because $h_{\min} \leq h(m-n)$ and $\hat{m}-1-n \geq m-n$ for all $m \in \{n+1, \ldots, \hat{m}-1\}$, (25) implies

$$\bar{\xi} - \alpha < \frac{h(m-n)}{m-n} \text{ for all } m \in \{n+1, \ldots, \hat{m}-1\}. \tag{27}$$

Multiplying (26) and (27) by $m-n$, the conjunction of these two inequalities implies that (23) is satisfied for $\bar{n} = n$ and $\bar{\xi}$.

It is straightforward to verify that (23) is equivalent to (21). ■

The sadistic conclusion is implied by a quasi-ordering R if there exist $\ell, n, m \in \mathbb{N}$, $u \in \mathbb{R}^\ell$, $v \in \mathbb{R}^n_{--}$, and $w \in \mathbb{R}^m_{++}$ such that $(u, v)P(u, w)$. Again, we negate this statement to obtain the axiom of avoidance of the sadistic conclusion.

Avoidance of the sadistic conclusion. For all $\ell, n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^\ell$, for all $v \in \mathbb{R}^n_{--}$, and for all $w \in \mathbb{R}^m_{++}$,

$$\neg\left((u, v)P(u, w)\right).$$

The sadistic conclusion can be avoided for positive and negative critical levels, in addition to the case in which α is equal to zero.

Theorem 6. *A threshold critical-level utilitarian quasi-ordering R satisfies avoidance of the sadistic conclusion if and only if*

$$h(k) \geq |\alpha| k \ \ \textit{for all } k \in \mathbb{N}. \tag{28}$$

Proof. Suppose that R is a threshold critical-level utilitarian quasi-ordering. Substituting, the sadistic conclusion is avoided if and only if

$$\sum_{i=1}^{n}[v_i - \alpha] - \sum_{i=1}^{m}[w_i - \alpha] < h(|n-m|) \text{ for all } n, m \in \mathbb{N}, \text{for all } v \in \mathbb{R}^n_{--}, \text{and for all } w \in \mathbb{R}^m_{++}; \tag{29}$$

again, we assume that $\rhd^{|n-m|} = \geq$ but the argument employed below works just as well for the case in which this threshold inequality is equal to $>$. It is immediate that we can, without loss of generality, assume that v and w are equal distributions; that is, there exist $\nu \in \mathbb{R}_{--}$ and $\mu \in \mathbb{R}_{++}$ such that $v = \nu \mathbf{1}_n$ and $w = \mu \mathbf{1}_m$. Substituting, it follows that (29) is equivalent to

$$n(\nu - \alpha) - m(\mu - \alpha) < h(|n - m|) \text{ for all } n, m \in \mathbb{N}, \text{for all } \nu \in \mathbb{R}_{--}, \text{and for all } \mu \in \mathbb{R}_{++}. \tag{30}$$

Next, we prove that (30) is equivalent to

$$\alpha(m - n) \leq h(|n - m|) \ \text{ for all } n, m \in \mathbb{N}. \tag{31}$$

Suppose first that (30) is satisfied. By way of contradiction, suppose that (31) is violated. Thus, there exist $n, m \in \mathbb{N}$ such that $\alpha(m-n) > h(|n-m|)$. Choosing $\nu \in \mathbb{R}_{--}$ and $\mu \in \mathbb{R}_{++}$ such that $\alpha(m - n) > \alpha(m - n) + n\nu - m\mu > h(|n - m|)$, it follows that

$$n(\nu - \alpha) - m(\mu - \alpha) < h(|n - m|),$$

contradicting (30).

Now suppose that (31) is satisfied. Because $n\nu < 0$ and $-m\mu < 0$, it follows that

$$n(\nu - \alpha) - m(\mu - \alpha) < \alpha(m - n) \leq h(|n - m|)$$

for all $n, m \in \mathbb{N}$, for all $\nu \in \mathbb{R}_{--}$, and for all $\mu \in \mathbb{R}_{++}$. Therefore, (30) is true.

Clearly, (31) is equivalent to $|\alpha|k \leq h(k)$ for all $k \in \mathbb{N}$, which is equivalent to (28). ■

Priority for lives worth living (Blackorby, Bossert, and Donaldson, 2005, Chapter 5) requires that a utility distribution in which everyone experiences a negative level of lifetime well-being cannot be better than a distribution in which all utilities are positive. In the context of the threshold critical-level utilitarian quasi-orderings, priority for lives worth living implies avoidance of the sadistic conclusion. This follows from the observation that these quasi-orderings satisfy the axiom of existence independence; see Blackorby, Bossert, and Donaldson (2005, Chapter 5).

The mere-addition principle is defined as follows.

Mere addition. For all $n, m \in \mathbb{N}$, for all $v \in \mathbb{R}^n$, and for all $w \in \mathbb{R}^m_{++}$,

$$\neg \left(vP(v,w) \right).$$

The following theorem identifies all threshold critical-level utilitarian quasi-orderings that satisfy mere addition.

Theorem 7. *A threshold critical-level utilitarian quasi-ordering R satisfies mere addition if and only if*

$$h(k) \geq \alpha k \;\; \textit{for all } k \in \mathbb{N}. \tag{32}$$

Proof. Suppose that R is a threshold critical-level utilitarian quasi-ordering. By definition, R satisfies mere addition if and only if, for all $n, m \in \mathbb{N}$, for all $v \in \mathbb{R}^n$, and for all $w \in \mathbb{R}^m_{++}$,

$$\sum_{i=1}^{n}[v_i - \alpha] - \sum_{i=1}^{n}[v_i - \alpha] - \sum_{i=1}^{m}[w_i - \alpha] < h(m) \tag{33}$$

where, again, we assume that $\rhd^m = \geq$; as is the case for the previous two theorems, the same proof applies if $\rhd^m = >$. Without loss of generality, we can assume that $w = \mu \mathbf{1}_m$ with $\mu \in \mathbb{R}_{++}$. Thus, (33) is satisfied if and only if

$$-m(\mu - \alpha) < h(m) \;\; \text{for all } m \in \mathbb{N} \text{ and for all } \mu \in \mathbb{R}_{++}. \tag{34}$$

We prove that (34) is equivalent to

$$m\alpha \leq h(m) \;\; \text{for all } m \in \mathbb{N}. \tag{35}$$

Suppose first that (34) is satisfied. By way of contradiction, suppose that (35) is violated. Thus, there exists $m \in \mathbb{N}$ such that $m\alpha > h(m)$. Choosing $\mu \in \mathbb{R}_{++}$ such that $m\alpha > m\alpha - m\mu > h(m)$, it follows that

$$-m(\mu - \alpha) > h(m),$$

contradicting (34).

Now suppose that (35) is satisfied. Because $-m\mu < 0$, it follows that

$$-m(\mu - \alpha) = -m\mu + m\alpha < m\alpha \leq h(m) \text{ for all } m \in \mathbb{N} \text{ and for all } \mu \in \mathbb{R}_{++}$$

and, therefore, (34) is true. Clearly, (35) is equivalent to (32). ■

By combining the observations of Theorems 5, 6, and 7, we can identify a necessary and sufficient condition for the compatibility of avoidance of the repugnant conclusion, avoidance of the sadistic conclusion, and the mere-addition principle for the general class of threshold critical-level utilitarian quasi-orderings.

Corollary 2. *A threshold critical-level utilitarian quasi-ordering R satisfies avoidance of the repugnant conclusion, avoidance of the sadistic conclusion, and mere addition if and only if*

$$\alpha > 0 \text{ and } h(k) \geq \alpha k \text{ for all } k \in \mathbb{N}$$

or

$$\alpha \leq 0 \text{ and } h(k) > -\alpha k \text{ for all } k \in \mathbb{N}.$$

An immediate consequence of this corollary is that any critical-band utilitarian quasi-ordering with a critical band $Q = [\alpha - \beta, \alpha + \beta]$ such that $\beta > |\alpha|$ satisfies the three desiderata considered in this section. Other examples that satisfy the conditions of the corollary include threshold critical-level utilitarian quasi-orderings such that $h(k) = \sqrt{k} + \beta k$ for all $k \in \mathbb{N}_0$, where $\beta > |\alpha|$.

7 Concluding remarks

This paper proposes a new possibility for introducing incompleteness in population ethics. Although the presence of non-comparabilities or incommensurabilities has long been recognized as a plausible phenomenon in complex comparisons that involve different population sizes, it appears that this issue has not been addressed in most of the existing literature. With the exception of the class of critical-band utilitarian orderings, we are not aware of any axiomatic approaches that do not assume complete goodness relations. We hope that our new class of quasi-orderings contributes towards the objective of filling this gap.

Whereas the quasi-orderings characterized in Theorems 3 and 4 exclude sequences of threshold inequalities that may differ across population-size differences, the general class axiomatized in Theorem 2 does not exhibit such a restriction. However, the subclasses that correspond to the cases in which all threshold inequalities are strict or all threshold inequalities are weak can be characterized by adding suitable continuity properties.

Consider first the critical-level utilitarian ordering. If, for all $k \in \mathbb{N}$, the sets $\{c \in \mathbb{R} \mid (u, c\mathbf{1}_k)Ru\}$ and $\{c \in \mathbb{R} \mid uR(u, c\mathbf{1}_k)\}$ are required to be closed in $\mathbb{R}$ for all $u \in \Omega$, it follows that R must be critical-level utilitarian. Analogously, the non-complete variant of this relation results if these sets are open.

If, for all $k \in \mathbb{N}$, the k-critical set contains at least two elements, the two distributions $u \in \Omega$ and $(u, c\mathbf{1}_k)$ must be either non-comparable according to R, or one of the two is better than the other. Thus, if the set $\{c \in \mathbb{R} \mid (u, c\mathbf{1}_k)Pu\}$ or the set $\{c \in \mathbb{R} \mid uP(u, c\mathbf{1}_k)\}$ is required to be open in $\mathbb{R}$ for all $u \in \Omega$, then the k-critical set must be closed and, analogously, if $\{c \in \mathbb{R} \mid (u, c\mathbf{1}_k)Pu\}$ or $\{c \in \mathbb{R} \mid uP(u, c\mathbf{1}_k)\}$ is closed in $\mathbb{R}$ for all $u \in \Omega$, it follows that the k-critical sets are open. In either case, all threshold inequalities must be the same for all positive population-size differences.

Appendix: Independence of the axioms in Theorem 2

To establish the independence of minimal increasingness, incremental equity, non-emptiness of k-critical sets, boundedness of k-critical sets, the symmetric critical-level population principle, k-critical-set independence, and k-critical-set singleton consistency, consider the following seven examples of quasi-orderings on Ω.

1. Define R^1 by letting, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uR^1v \;\Leftrightarrow\; \sum_{i=1}^{n} u_i \leq \sum_{i=1}^{m} v_i.$$

The (quasi-)ordering R^1 satisfies all axioms in the statement of Theorem 2, other than minimal increasingness.

2. Let $g\colon \mathbb{R} \to \mathbb{R}$ be an increasing and strictly concave function. Define R^2 by letting, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uR^2v \;\Leftrightarrow\; \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{m} g(v_i)$$

This (quasi-)ordering satisfies all axioms in the statement of Theorem 2, other than incremental equity.

3. Define R^3 by letting, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uR^3v \;\Leftrightarrow\; \left[n = m \;\text{ and }\; \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i \right] \;\text{ or }\; n < m.$$

The (quasi-)ordering R^3 satisfies all axioms in the statement of Theorem 2, other than non-emptiness of k-critical sets.

4. Define R^4 by letting, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uR^4v \;\Leftrightarrow\; n = m \;\text{ and }\; \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i.$$

The quasi-ordering R^4 satisfies all axioms in the statement of Theorem 2, other than boundedness of k-critical sets.

5. Define R^5 by letting, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uR^5v \;\Leftrightarrow\; \left[n = m \text{ and } \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i\right] \text{ or } \left[n \neq m \text{ and } \sum_{i=1}^{n} [u_i - c] > \sum_{i=1}^{m} [v_i - c] \text{ for all } c \in (1, 2]\right].$$

The quasi-ordering R^5 satisfies all axioms in the statement of Theorem 2, other than the symmetric critical-level population principle. That the symmetric critical-level population principle is violated follows from the observation that the k-critical set $\bar{Q}^k$ for R^5 is given by the half-open interval $(1, 2]$.

6. Define R^6 by letting, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uR^6v \;\Leftrightarrow\; \frac{1}{n}\sum_{i=1}^{n} u_i \geq \frac{1}{m}\sum_{i=1}^{m} v_i.$$

The quasi-ordering R^6 satisfies all axioms in the statement of Theorem 2, other than k-critical-set independence.

7. Define R^7 by letting, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^m$,

$$uR^7v \;\Leftrightarrow\; \left[n = m \text{ and } \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i\right] \text{ or } \left[n \neq m \text{ and } n, m \in \{1, 2\} \text{ and } \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i\right] \text{ or } \left[n \neq m, \text{ and } [n \notin \{1, 2\} \text{ or } m \notin \{1, 2\}] \text{ and } \sum_{i=1}^{n} u_i > \sum_{i=1}^{m} v_i\right].$$

The quasi-ordering R^7 satisfies all axioms in the statement of Theorem 2, other than k-critical-set singleton consistency. To see that the latter axiom is violated, observe that $\bar{Q}^k = \{0\}$ for all $k \in \mathbb{N}$. By definition, $(0, 0)I(0)$ and $(0, 0, 0)N(0, 0)$, in contradiction to k-critical-set singleton consistency.

To prove that R^7 is transitive, suppose that $n, m, \ell \in \mathbb{N}$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, and $w \in \mathbb{R}^\ell$ are such that uR^7v and vR^7w. Clearly,

$$\sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i \text{ and } \sum_{i=1}^{m} v_i \geq \sum_{i=1}^{\ell} w_i \tag{36}$$

and, therefore,

$$\sum_{i=1}^{n} u_i \geq \sum_{i=1}^{\ell} w_i. \tag{37}$$

If ($n = m$ or $m = \ell$ or $n = \ell$) and one of the inequalities in (36) is strict, the inequality in (37) must be strict and, by definition, we obtain uR^7w.

If ($n = m$ or $m = \ell$ or $n = \ell$) and both of the inequalities in (36) are weak, it follows that $n = \ell$ or none of n, m, and ℓ is greater than 2. In both cases, the inequality in (37) must be weak and, by definition, this is sufficient to conclude that uR^7w.

If the three population sizes n, m, and ℓ are pairwise distinct and any of them is greater than 2, it follows that at least one of the inequalities in (36) is strict and, therefore, the inequality in (37) is strict. The definition of R^7 implies that uR^7w.

If the three population sizes n, m, and ℓ are pairwise distinct and all three of them are less than or equal to 2, it follows that all inequalities in (36) and in (37) are weak, which implies that uR^7w by definition.

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