The structure of critical sets

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The structure of critical sets*

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Abstract. The purpose of this paper is to address some ambiguities and misunderstandings that appear in previous studies of population ethics. In particular, we examine the structure of intervals that are employed in assessing the value of adding people to an existing population. Our focus is on critical-band utilitarianism and critical-range utilitarianism, which are commonly-used population theories that employ intervals, and we show that some previously assumed equivalences are not true in general. The possible discrepancies can be attributed to the observation that critical bands need not be equal to critical sets. The critical set for a moral quasi-ordering is composed of all utility numbers such that adding someone with a utility level in this set leads to a distribution that is not comparable to the original (non-augmented) distribution. The only case in which critical bands and critical sets coincide obtains when the critical band is an open interval. In this respect, there is a stark contrast between critical-band utilitarianism and critical-range utilitarianism: the critical set that corresponds to a critical-range quasi-ordering always coincides with the interval that is used to define the requisite quasi-ordering. As a consequence, an often presumed equivalence of critical-band utilitarianism and critical-range utilitarianism is not valid unless, again, the critical band and the critical range (and, consequently, the requisite critical sets) are given by the same open interval.

1 Introduction

The question of why and how non-comparabilities and incommensurabilities arise remains a central challenge in ethics; see, for example, Levi (1986) and Chang (1997, 2015). While various explanations have been proposed, an extensively debated issue continues to be the existence and the identity of persons. When the number and the identities of those in existence differ across alternative states of affairs, these states often are non-comparable or incommensurable. The resulting lack of comparability typically stems from two fundamental difficulties: determining what constitutes a valuable life, and comparing the welfare of distinct individuals in a meaningful way; see Parfit (2016), Nebel (2022), and Rabinowicz (2022).

This paper examines two specific perspectives in population ethics—critical-band utilitarianism and critical-range utilitarianism—which have generated controversy since the earliest discussions of the presence of non-comparabilities. According to these views, an individual joining the world is not necessarily undesirable, even if it is not necessarily desirable or equally good either. Rather, these theories propose that there is an interval of well-being levels that creates non-comparability in evaluating the value of existence. Consider a scenario where an individual with a particular level of well-being joins the world, resulting in non-comparability. Would an individual with a well-being level very close to this level produce the same result? This question directly relates to whether the relevant interval is open or closed. Given the complexity of the literature that evolved throughout decades, a formal analysis is required to clarify these issues. Our purpose is to addresses this topic, which appears to have been overlooked so far in spite of its significance.

Section 2 explains the background of this paper and the main concepts that we examine. In Section 3, we present our definitions and notational conventions. In addition, we introduce two types of utilitarian theories that allow for non-comparabilities—namely, critical-band utilitarianism and critical-range utilitarianism. Our main results are presented in Section 4, where we examine some logical relationships within and between the critical-band utilitarian and the critical-range utilitarian theories. Section 5 contains a discussion of critical sets. In particular, we establish a variant of a result by Blackorby, Bossert, and Donaldson (1996, 2005). This observation demonstrates that, under some plausible conditions, the critical set for a moral quasi-ordering (a reflexive and transitive relation that need not be complete) must be an interval. Section 6 concludes. The proofs of all formal results are collected in Appendix A. In Appendix B, we show that our definition of critical-range utilitarianism is equivalent to that of Rabinowicz (2009).

2 Critical-band utilitarianism, critical-range utilitarianism, and critical sets

In population ethics, the central issue is how to assess the comparative goodness of states of affairs when the size and the composition of the population may change. If a welfarist position is adopted, these states can be compared by establishing a goodness relation that is capable of comparing the distributions of lifetime well-being that obtain in each state. We use the terms utility, well-being, and lifetime well-being interchangeably. Each individual's well-being is represented by a numerical value.

The notion of neutrality constitutes an important benchmark. A life, taken as a whole, is a neutral life if it is, from the viewpoint of the person leading it, neither better nor worse than a life without any experiences. We note that there are alternative accounts of neutrality but, because this choice does not affect the observations reported in this paper, any of these alternative options would do just as well. Following the standard convention in population ethics, we normalize the utility level that represents a neutral life to zero. We assume that, with this normalization, individual levels of lifetime well-being are numerically significant.

A critical level is, in general, distinct from a neutral level of utility. If an individual whose utility is at the critical level is added to a given population, the resulting distribution is, according to the moral goodness relation, neither better nor worse than the original, provided that the utility levels of those who exist in both situations are unaffected by this change. A critical level need not exist and if it exists, it need not be unique. Moreover, critical levels may depend on the utility distribution under consideration. We emphasize that a critical level need not be equal to a level that represents a neutral life.

In his monograph *Weighing Lives*, Broome (2004) describes what he calls the intuition of neutrality. In particular, he writes (Broome, 2004, p. 143),

> "We think intuitively that adding a person to the world is very often ethically neutral. We do not think that just a single level of wellbeing is neutral."

He then continues (Broome, 2004, pp. 145–146),

> "Interpreted axiologically, in terms of goodness, the intuition is that if a person is added to the population of the world, her addition has no positive or negative value in itself."

It is important to note that Broome seems to use the expression "ethically neutral" when referring to moral rather than individual or prudential goodness and, therefore, according to our convention, he is making a statement about critical levels. This intuition is often considered plausible and has been discussed by Qizilbash (2005, 2007), Rabinowicz (2009, 2012, 2022), and Gustafsson (2020).

Broome's idea can be linked to theories in which the addition of a person may lead to a utility distribution that is not better than, not worse than, and not as good as the original distribution. Such a theory deviates from the traditional framework where all distributions of lifetime well-being can be ranked—goodness relations may be incomplete, making non-comparability a possibility. A population theory of this nature is proposed by Blackorby and Donaldson (1992) in their comment on a criticism of Broome (1992) that was directed towards their theory of critical-level utilitarianism (Blackorby and Donaldson, 1984). The critical-level utilitarian orderings use the sum of the differences between individual utilities and a fixed critical level as the criterion to rank any two utility distributions. If the critical level is equal to zero (the level that represents neutrality), classical or total utilitarianism results. Once completeness is no longer required as a property of a moral goodness relation, a critical level may be replaced by an interval of utility levels. Perhaps the most commonly discussed theory that is based on intervals is what Blackorby, Bossert, and Donaldson (1996, 1997) initially refer to as incomplete critical-level utilitarianism. Later on (Blackorby, Bossert, and Donaldson, 2005), they adopt the label critical-band utilitarianism, first employed by Broome (1996).

Critical-band utilitarianism declares a distribution morally at least as good as another if the former is at least as good as the latter according to critical-level utilitarianism for all values in the interval that represents the critical band. If the interval is non-degenerate (that is, it contains more than one number), critical-band utilitarianism does not generate an ordering but merely a quasi-ordering because some utility distributions are declared non-comparable.

There is a subtle difference between critical-band utilitarianism and a related theory examined by Qizilbash (2005, 2007), Rabinowicz (2009, 2012, 2022), Gustafsson (2020), and Williamson (2021); Gustafsson (2020) and Williamson (2021) label this theory critical-range utilitarianism. Whereas critical-band utilitarianism performs different-number comparisons by declaring a utility distribution morally at least as good as another whenever the former is at least as good as the latter according to the critical-level utilitarian criterion for all numbers in the critical band, critical-range utilitarianism replaces the at-least-as-good-as requirement with betterness. To be precise, critical-range utilitarianism declares a distribution morally better than another if the former is better than the latter according to critical-level utilitarianism for all values in the interval. Seemingly, this difference is frequently ignored because of a belief that the two formulations are equivalent. One of the main objectives of our paper is to show that these theories are actually different.

We stress that the difference between critical-band utilitarianism and critical-range utilitarianism is not rooted in any difference in the types of sets that are employed. Rather, it is the difference between at-least-as-goodness and betterness applied to each number in the set that distinguishes the two theories. The reason why we employ these labels is that, in an effort to avoid confusion, we want to respect the terms that have been used in the earlier literature.

A related widespread misconception that we address here is that the numbers in a critical band have the property that, if an individual with a lifetime level of well-being within the critical band is added to a given distribution, the resulting distribution and the original are non-comparable. While this is true for all numbers located in the interior of the critical band, it is not correct for endpoints. Because of these subtle distinctions that need to be made in order to clarify these issues, great care must be taken regarding the choice of the terms that we employ for some similar but ultimately distinct entities.

Consider a quasi-ordering that is used to compare utility distributions. The critical set corresponding to this quasi-ordering is the set of all utility numbers such that, if a person with a utility level in this set is added to a given utility distribution, the resulting augmented distribution and the original are non-comparable. As we shall demonstrate, the critical set for a critical-band utilitarian quasi-ordering is not always equal to the critical band that is employed in the definition of this quasi-ordering, a distinction that numerous earlier contributions failed to make. This imprecision goes back as far as Blackorby, Bossert, and Donaldson (1996, 2005) who, on some occasions, seem to suggest that the critical band consists of all levels of well-being that lead to non-comparability when experienced by an additional person. In contrast, the critical set that corresponds to a critical-range utilitarian quasi-orderings turns out to be equal to the critical range. This observation highlights the difference between the two classes of quasi-orderings discussed here.

To reiterate, we use the term critical set for the set of utility levels such that the addition of a person who experiences a level of lifetime well-being given by any value within this set leads to a distribution that is not comparable to the original. This critical set is to be distinguished from the critical band that is used in the definition of a critical-band utilitarian quasi-ordering: adding a person who experiences a level of lifetime well-being that is located at one of the endpoints of the critical band does not lead to a distribution that is incomparable to the initial distribution. In fact, if the upper (lower) endpoint is included in a critical band, adding someone at this level leads to a better (worse) distribution. Thus, the only scenario in which the critical set and the critical band coincide obtains if the critical band is an open interval—that is, its two endpoints are not included. If the critical band is closed or half-open (that is, it contains at least one of its two endpoints), the critical set is a strict subset of the critical band. The critical set and the critical range, however, always coincide, no matter whether the critical range is open, half-open, or closed.

3 Moral quasi-orderings

As alluded to in the previous section, we operate in the standard framework of welfarist population ethics. That is, we assess states of the world with possibly different populations by establishing a moral goodness relation defined on distributions of lifetime well-being.

A typical well-being distribution is given by a vector, such as $u = (u_1, \dots, u_n)$, where u_i is the utility level of individual i and n is the population size. The set Ω of all possible utility distributions collects all utility distributions u of all positive finite population sizes n. Different distributions $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ in Ω may have different population sizes—that is, n need not be equal to m. We ignore the identities of individuals in utility distributions. Therefore, even when comparing the same-number utility distributions $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$, u_i and v_i may represent the utility levels of different individuals. This means that our approach respects the principle of anonymity.

A life, taken as a whole, is neutral if it is, from the viewpoint of the person leading it, as good as a life without any experiences. Other accounts of neutrality have been proposed but, for our purposes, it does not matter which of them applies; our observations are independent of this choice. Following the standard convention in population ethics, we normalize the individual utilities so that neutrality is represented by a utility level of zero.

Moral judgments are represented by a goodness relation R defined on the set Ω of utility distributions. Because R is interpreted as a goodness relation, uRv means that u

is considered at least as good as v. As usual, the betterness relation P and the equal-goodness relation I corresponding to R are defined as

$$uPv \ \Leftrightarrow\ uRv \text{ and } \neg vRu$$

and

$$uIv \ \Leftrightarrow\ uRv \text{ and } vRu.$$

Thus, uPv is interpreted as u is better than v, and uIv means that u and v are equally good. It is easy to see that uRv holds if and only if either uPv or uIv holds.

A critical level of utility is an attribute of a moral goodness relation. If an individual whose lifetime well-being is at the critical level is added to a given utility distribution, the augmented distribution is, according to the moral goodness relation, neither better nor worse than the original. A critical level need not be equal to a level that represents a neutral life.

The standard assumption in the literature is that R is an ordering—a reflexive, complete, and transitive relation. Reflexivity means that every distribution u is at least as good as itself (that is, uRu), completeness demands that any two distinct distributions u and v can be compared (that is, we have uRv or vRu whenever u is not equal to v), and transitivity is the usual coherence requirement that, whenever a distribution u is at least as good as a distribution v (that is, uRv) and v is at least as good as a distribution w (that is, vRw), then u must be at least as good as w (that is, uRw).

This paper focuses on moral goodness relations that allow for non-comparable utility distributions. As a result, the relations that we consider are quasi-orderings—reflexive and transitive relations that need not be complete. We write uNv when two distributions u and v are non-comparable, that is,

$$uNv \ \Leftrightarrow\ \neg uRv \text{ and } \neg vRu.$$

Clearly, an ordering is a complete quasi-ordering.

An example of a goodness relation that is an ordering is classical (or total) utilitarianism. In this case, the relation R is defined by letting, for all population sizes n and m and for all utility distributions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$,

$$uRv \ \Leftrightarrow\ \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i.$$

This ordering is the cornerstone of one of the most well-established theories in ethics and welfare economics. However, as first pointed out by Parfit (1976, 1982, 1984), utilitarianism is afflicted with a fundamental difficulty. This is because it implies what Parfit refers to as the repugnant conclusion. The repugnant conclusion is implied if, for any arbitrarily large population of size n in which everyone enjoys an arbitrarily high level of well-being ξ, and for any utility level ε above neutrality but arbitrarily close to it, there exists a larger population size $m > n$ such that everyone alive experiences a lifetime level of well-being ε, and the latter distribution is considered better than the former. In other words, if the repugnant conclusion obtains, mass poverty can be used to substitute for quality of life, no matter how close to neutrality everyone's utility may be. The fundamental problem with classical utilitarianism is that its critical level coincides with the level that represents a neutral life.

A possible way to avoid the repugnant conclusion is proposed by Blackorby and Donaldson (1984). Critical-level utilitarianism is a class of goodness orderings that generalizes total utilitarianism by allowing a fixed critical level to diverge from the level that represents a neutral life. Suppose that this critical level is given by the number α, which may be positive, equal to zero (the level that represents neutrality), or negative. Critical-level utilitarianism with a critical level α is defined by letting, for all population sizes n and m, and for all utility distributions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$,

$$uRv \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - \alpha] \geq \sum_{i=1}^{m}[v_i - \alpha].$$

It is immediate that total utilitarianism results if α is equal to zero—and, therefore, the repugnant conclusion is entailed. The repugnant conclusion is also implied for negative values of the critical level α. For positive critical levels, however, the repugnant conclusion is avoided. See Blackorby and Donaldson (1984) for a detailed discussion of critical-level utilitarianism.

In the case of critical-level utilitarianism, there is a single utility number that corresponds to a critical level. This feature is criticized by Broome (1992), who argues that a single critical level is difficult to identify and justify. In their comment on Broome (1992), Blackorby and Donaldson (1992) propose an incomplete variant of critical-level utilitarianism that employs a (non-degenerate and bounded) interval to define a class of moral quasi-orderings.

The theory outlined by Blackorby and Donaldson (1992) is formally developed and discussed in a series of contributions by Blackorby, Bossert, and Donaldson (1996, 1997, 2005). They propose the class of critical-band utilitarian quasi-orderings, initially labeled incomplete critical-level utilitarianism in Blackorby, Bossert, and Donaldson (1996, 1997). A goodness relation R is critical-band utilitarian if there exists a non-degenerate and bounded interval Q such that, for all population sizes n and m and for all utility distributions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$,

$$\begin{aligned} uRv \;\;\Leftrightarrow\;\; & \left[n = m \;\text{ and }\; \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i \right] \;\text{ or} \\ & \left[n \neq m \;\text{ and }\; \sum_{i=1}^{n}[u_i - c] \geq \sum_{i=1}^{m}[v_i - c] \;\text{ for all } c \in Q \right]. \end{aligned} \tag{1}$$

A non-degenerate interval is a set Q of real numbers that includes at least two elements and, for any two $a, b \in Q$ such that $a < b$, the number c is in Q for all c such that $a < c < b$. Boundedness of Q means that there exist numbers $\underline{K}$ and $\overline{K}$ such that $\underline{K} < c < \overline{K}$ for all c in Q—that is, the values in the interval cannot be unboundedly low or unboundedly high. Whenever appropriate to avoid ambiguities, we use R_b^Q to denote the critical-band utilitarian quasi-ordering associated with the non-degenerate and bounded interval Q. Clearly, a critical-band utilitarian quasi-ordering is not complete because a distribution u with n people may be at least as good as a distribution v with $m \neq n$ people according to critical-level utilitarianism with a critical level c in Q, but v may be better than u according to critical-level utilitarianism with another critical level c' in Q.

Observe that, according to critical-band utilitarianism, a utility distribution is at least as good as another if the former is at least as good as the latter for all critical-level utilitarian relations associated with the critical band. That is, the critical-band criterion is defined in terms of at-least-as-goodness relations—and not in terms of betterness relations. As we explain later, replacing the weak inequality with a strict inequality in (1) leads, in general, to a different class of quasi-orderings. This clarification is of importance because there appears to be a widely held misperception to the effect that the two formulations are interchangeable. The ambiguity can be traced back to Blackorby, Bossert, and Donaldson (1996) who use at-least-as-goodness on some occasions but betterness on others to describe critical-band utilitarianism.

Notably, when two populations with different sizes are compared by critical-band utilitarianism, there is no possibility of equal goodness. Different-number comparisons of critical-band utilitarianism are defined by the second line of (1), which means that u is morally at least as good as v if and only if u is at least as good as v according to all critical-level utilitarian relations defined with critical levels $c \in Q$, the critical band. Therefore, if the critical-band utilitarian criterion declares equal goodness for two distributions u and v with different population sizes, it must be true that

$$\left[n \neq m \;\text{ and }\; \sum_{i=1}^{n} [u_i - c] = \sum_{i=1}^{m} [v_i - c] \;\text{ for all } c \in Q \right].$$

However, this is impossible because the requisite equality cannot be satisfied by more than one value of c and, therefore, it cannot be satisfied for all values in Q; note that this interval is assumed to be non-degenerate—that is, it cannot contain a single number only. In sum, there are only two possibilities in different-number comparisons of critical-band utilitarianism: either betterness or non-comparability obtains.

Population theories that are based on intervals have been discussed in numerous subsequent contributions, including those of Qizilbash (2005, 2007), Rabinowicz (2009, 2012, 2022), and Gustafsson (2020). Rabinowicz proposes a theory that can, in part, be described as follows. There exists a non-degenerate and bounded interval Q such that, for all population sizes n and m and for all utility distributions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$,

$$\begin{aligned} uRv \;\Leftrightarrow\; & \left[n = m \;\text{ and }\; \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i \right] \;\text{ or} \\ & \left[n \neq m \;\text{ and }\; \sum_{i=1}^{n} [u_i - c] > \sum_{i=1}^{m} [v_i - c] \;\text{ for all } c \in Q \right]. \end{aligned}$$

We refer to these criteria as critical-range utilitarian quasi-orderings. Observe that they differ from the critical-band utilitarian quasi-orderings in that a weak inequality is replaced by a strict inequality when defining different-number comparisons. Although this difference is not rooted in any divergence of the intervals Q being used but, rather, in the type of inequality that applies, we choose to distinguish the two by means of the terms critical band versus critical range. Our motivation is to respect the nomenclature that developed in the literature but we stress that the distinction rests on the different inequalities being used. With this caveat in mind, we think that no ambiguities should arise.

In analogy to the critical-band utilitarian quasi-orderings, we write R_r^Q for the critical-range utilitarian quasi-ordering associated with the non-degenerate and bounded interval Q.

We note that Rabinowicz (2009, p. 404; 2022, p. 122) uses a slightly different formulation of this theory. Two of his postulates are expressed as

(i) u is better than v if and only if $\sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c]$ for all $c \in Q$;

(ii) u is as good as v if and only if $\sum_{i=1}^{n}[u_i - c] = \sum_{i=1}^{m}[v_i - c]$ for all $c \in Q$,

and this is indeed equivalent to our definition of critical-range utilitarianism. The reason is that part (ii) can only ever apply to same-number comparisons; again, if $n \neq m$, the requisite equality cannot be satisfied for all values in the non-degenerate interval Q. A formal proof of the equivalence of the two definitions is provided in Appendix B.

Rabinowicz labels his proposal neutral-range utilitarianism (Rabinowicz, 2009, 2012, 2022) because his interpretation of the interval is that it represents levels of neutrality rather than critical levels. The term critical-range utilitarianism is used by Gustafsson (2020) and Williamson (2021); see also Thomas (2023).

It seems to be assumed (at least implicitly) in numerous contributions that critical-range utilitarianism is identical to critical-band utilitarianism; see also our earlier remarks to the effect that weak and strict inequalities are used interchangeably by Blackorby, Bossert, and Donaldson (1996). As another instance, note that Rabinowicz (2022, p. 123) writes (see also Rabinowicz, 2009, p. 403),

> "This axiology is formally identical with the theory that has been put forward by Blackorby, Bossert and Donaldson (1996)."

Qizilbash (2005, 2007), Gustafsson (2020), and Williamson (2021) also claim that critical-range utilitarianism is proposed by Blackorby, Bossert, and Donaldson (1996, 1997). Statements of this nature are correct only if critical-range utilitarianism is identical to critical-band utilitarianism. As we demonstrate in the following section, this is not the case in general but only under the additional assumption that the interval Q be open.

4 Equivalences and non-equivalences

The notion of a critical set is an essential ingredient when analyzing quasi-orderings in population ethics. Following Blackorby, Bossert, and Donaldson (1996, 2005), we define the critical set $CS(u)$ of a utility distribution u for a moral quasi-ordering R as the set of utility levels that, if experienced by an additional person, lead to a utility distribution that is not comparable to the original distribution u according to R. Thus, this critical set is defined as

$$CS(u) = \{c \in \mathbb{R} \mid (u, c)Nu\},$$

where $\mathbb{R}$ is the set of all possible utility values, given by the set of all real numbers, and (u, c) is the augmented distribution; that is, $(u, c) = (u_1, \ldots, u_n, c)$ if $u = (u_1, \ldots, u_n)$.

In general, critical sets can depend on the utility distribution under consideration, as observed by Blackorby, Bossert, and Donaldson (1996, 2005). However, in analogy to Blackorby, Bossert, and Donaldson (1996, 2005), we assume that these critical sets do not depend on the utility distribution under consideration; that is, that there exists a set C such that

$$C = CS(u) \ \text{ for all } u \in \Omega.$$

This assumption of critical-set independence is fairly innocuous in the context of quasi-orderings that are based on utilitarian considerations. This is the case because the quasi-orderings that we examine in this paper satisfy existence independence, which is a variable-population version of separability. Existence independence requires that, for any three utility distributions u, v, and w, augmenting u and v by w does not change the relative goodness of u and v—that is, u is at least as good as v if and only if (u, w) is at least as good as (v, w). In other words, the (non)-existence of those associated with the distribution w does not affect the relative ranking of the distributions u and v according to R. As a consequence of existence independence, the critical sets for a quasi-ordering R are independent of the utility distributions to be compared once a person is added to the population. Thus, the critical set $CS(u)$ is invariant across utility distributions. Blackorby, Bossert, and Donaldson (1996, 2005) impose a weaker property that still implies distribution-invariance of the critical sets. Existence independence has gained prominence in the recent literature on population ethics (and on the notion of fanaticism) due to its analytical tractability and intuitive appeal when comparing populations of different sizes; see, for example, Goodsell (2021), Thomas (2023), and Russell (2024).

If R_b^Q (resp. R_r^Q) is the critical-band (resp. critical-range) utilitarian quasi-ordering associated with the non-degenerate and bounded interval Q, the critical set for R_b^Q (resp. R_r^Q) is denoted by C_b^Q (resp. C_r^Q).

A few further definitions are required for the statements of our results. Given a non-degenerate and bounded interval Q with endpoints $\underline{c}$ and $\overline{c}$ such that $\underline{c} < \overline{c}$, let $\inf Q = \underline{c}$, $\sup Q = \overline{c}$, and $\operatorname{int} Q = \{c \in \mathbb{R} \mid \underline{c} < c < \overline{c}\}$ denote the infimum, the supremum, and the interior of Q, respectively. The infimum is the lower endpoint of Q, the supremum is the upper endpoint of Q, and the interior is composed of all points strictly between the two endpoints. Note that Q may or may not contain one or both of its endpoints—that is, Q may be equal to the closed interval $[\underline{c}, \overline{c}] = \{c \in \mathbb{R} \mid \underline{c} \leq c \leq \overline{c}\}$, the open interval $(\underline{c}, \overline{c}) = \{c \in \mathbb{R} \mid \underline{c} < c < \overline{c}\}$, or one of the half-open intervals $[\underline{c}, \overline{c}) = \{c \in \mathbb{R} \mid \underline{c} \leq c < \overline{c}\}$ and $(\underline{c}, \overline{c}] = \{c \in \mathbb{R} \mid \underline{c} < c \leq \overline{c}\}$.

Our first result establishes an equivalent formulation of critical-band utilitarianism.

Theorem 1. *Let Q be a non-degenerate and bounded interval. For all population sizes n and m, for all utility distributions $u = (u_1, \ldots, u_n)$, and for all utility distributions $v = (v_1, \ldots, v_m)$, $uR_b^Q v$ if and only if*

(i) $n = m$ *and* $\sum_{i=1}^{n} u_i \geq \sum_{i=1}^{n} v_i$;

or

(ii) $n > m$ *and*

$$\sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c] \text{ for } c = \inf Q \text{ and for all } c \in \operatorname{int} Q \text{ and} \tag{2}$$

$$\sum_{i=1}^{n}[u_i - c] \geq \sum_{i=1}^{m}[v_i - c] \text{ for } c = \sup Q; \tag{3}$$

or

(iii) $n < m$ *and*

$$\sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c] \text{ for } c = \sup Q \text{ and for all } c \in \operatorname{int} Q \text{ and} \tag{4}$$

$$\sum_{i=1}^{n}[u_i - c] \geq \sum_{i=1}^{m}[v_i - c] \text{ for } c = \inf Q. \tag{5}$$

Thus, critical-band utilitarianism declares one distribution to be better than another with a different population size if the former is better than the latter according to critical-level utilitarianism for all elements of the critical band Q, except for the endpoints of Q. This is the case even though the definition of these quasi-orderings is based on at-least-as-goodness relations. Also, Theorem 1 implies that the lower and upper endpoints of a critical band Q play a crucial role even if Q is not a closed interval—that is, even if Q does not contain one or both of the endpoints.

To illustrate, suppose that the population size of u is larger than that of v (that is, $n > m$). In this case, critical-band utilitarianism declares u to be least as good as v if and only if the former is better than the latter for all critical-level utilitarian orderings with a critical level in the half-open interval $[\underline{c}, \overline{c})$, and the former is at least as good as the latter for the critical-level utilitarian ordering with the critical level $\overline{c}$. Therefore, comparisons that involve the endpoints $\underline{c}$ and $\overline{c}$ matter.

Theorem 1 can be employed to obtain the following equivalence result regarding critical-band utilitarian quasi-orderings.

Theorem 2. *Suppose that $\underline{c}, \overline{c} \in \mathbb{R}$ are such that $\underline{c} < \overline{c}$. Then*

$$R_b^{[\underline{c},\overline{c}]} = R_b^{[\underline{c},\overline{c})} = R_b^{(\underline{c},\overline{c}]} = R_b^{(\underline{c},\overline{c})}.$$

This theorem states that a critical-band utilitarian quasi-ordering is completely determined by the endpoints of its critical band; whether one or both of these endpoints are members of the interval is of no relevance because all four of the associated quasi-orderings are identical. This is the case because the critical sets for all of the four possible intervals coincide and are given by the corresponding open interval—that is,

$$C_b^{[\underline{c},\overline{c}]} = C_b^{[\underline{c},\overline{c})} = C_b^{(\underline{c},\overline{c}]} = C_b^{(\underline{c},\overline{c})} = (\underline{c},\overline{c})$$

for all $\underline{c}, \overline{c} \in \mathbb{R}$ such that $\underline{c} < \overline{c}$.

As an example, consider the critical-band utilitarian quasi-ordering $R_b^{[0,2]}$ associated with the critical band $Q = [0, 2]$. According to our results, the critical set for this quasi-ordering is given by $C_b^{[0,2]} = (0, 2)$. Suppose that $u = (u_1, \ldots, u_n)$ is a utility distribution with population size n. It follows that

$$\begin{aligned} (u,2)R_b^{[0,2]}u \;\; &\Leftrightarrow \;\; \sum_{i=1}^{n}[u_i - c] + (2-c) \geq \sum_{i=1}^{n}[u_i - c] \;\text{ for all } c \in [0,2] \\ &\Leftrightarrow \;\; 2 \geq c \;\text{ for all } c \in [0,2]. \end{aligned} \tag{6}$$

The inequality in (6) is always true and, therefore, it follows that $(u, 2)R_b^{[0,2]}u$. As a consequence, the endpoint 2 cannot belong to the critical set $C_b^{[0,2]}$. That the endpoint 0 cannot be a member of $C_b^{[0,2]}$ can be shown analogously. This suggests that considering open intervals is natural for the case of critical-band utilitarianism because this is the only case in which the critical band and the critical set coincide.

In contrast, the critical set for a critical-range utilitarian quasi-ordering always coincides with the interval that defines this quasi-ordering. This observation applies to all four types of intervals. Therefore, there is a marked difference between critical-band utilitarianism and critical-range utilitarianism because, in the former case, this equivalence only obtains in the case of an open interval.

Theorem 3. *Suppose that Q is a non-degenerate and bounded interval. Then $C_r^Q = Q$.*

Theorem 3 shows that the critical set C_r^Q for the critical-range utilitarian quasi-ordering R_r^Q associated with a non-degenerate and bounded interval Q is equal to Q itself, no matter whether Q is closed, half-open, or open. This implies that there are some subset relationships between the four critical-range quasi-orderings $R_r^{[\underline{c},\overline{c}]}$, $R_r^{[\underline{c},\overline{c})}$, $R_r^{(\underline{c},\overline{c}]}$, and $R_r^{(\underline{c},\overline{c})}$ for any two endpoints $\underline{c}$ and $\overline{c}$ with $\underline{c} < \overline{c}$.

For instance, consider the critical ranges $[\underline{c}, \overline{c}]$ and $[\underline{c}, \overline{c})$. Because the critical set for a critical-range utilitarian quasi-ordering coincides with the critical range that defines this relation, it follows that distributions of the type $(u, \overline{c})$ and u can be compared according to $R_r^{[\underline{c},\overline{c})}$ but not according to $R_r^{[\underline{c},\overline{c}]}$—the number $\overline{c}$ is in the critical set $C_r^{[\underline{c},\overline{c}]}$ but not in $C_r^{[\underline{c},\overline{c})}$. As a consequence, $(u, \overline{c})$ is better than u according to $R_r^{[\underline{c},\overline{c})}$ but not according to $R_r^{[\underline{c},\overline{c}]}$: because the critical set for the latter quasi-ordering contains $\overline{c}$, the two distributions are non-comparable. In general, whenever a critical range Q is a strict subset of a critical range Q', it follows immediately that the critical-range utilitarian quasi-ordering R_r^Q is a strict superset of the critical-range quasi-ordering $R_r^{Q'}$ because more pairs of distributions can be compared according to the former. Thus, we obtain the subset relationships

$$R_r^{[\underline{c},\overline{c}]} \subsetneq R_r^{(\underline{c},\overline{c}]} \subsetneq R_r^{(\underline{c},\overline{c})} \;\text{ and }\; R_r^{[\underline{c},\overline{c}]} \subsetneq R_r^{[\underline{c},\overline{c})} \subsetneq R_r^{(\underline{c},\overline{c})}.$$

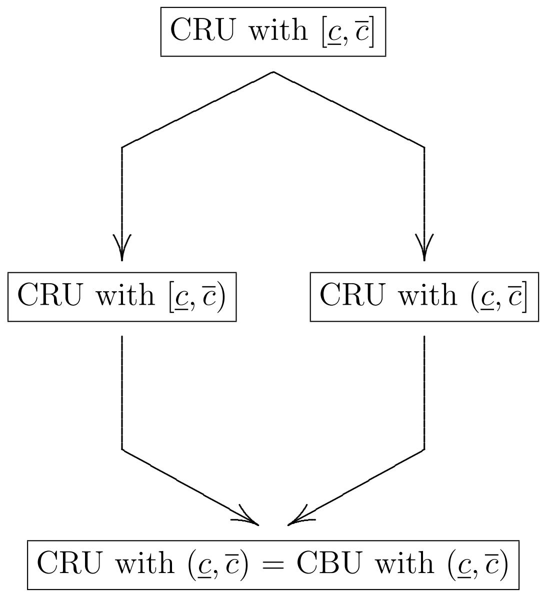

Figure 1: Relationships between critical-range utilitarian and critical-band utilitarian theories

There is no subset relationship between the two quasi-orderings $R_r^{[\underline{c},\overline{c})}$ and $R_r^{(\underline{c},\overline{c}]}$ because distributions that involve the addition of the lower endpoint $\underline{c}$ to a given distribution u can be compared according to the latter relation but not according to the former, and the reverse is true for the addition of the upper endpoint $\overline{c}$ to a distribution u.

As a consequence of the above observations, it follows that critical-band utilitarianism coincides with critical-range utilitarianism only if the interval that identifies the critical band and the critical range is open—in each of the three remaining cases, fewer comparisons are possible according to critical-range utilitarianism. We summarize the results of this section in the following corollary that combines all of our observations regarding the critical sets for critical-band utilitarianism and for critical-range utilitarianism. A diagrammatic illustration of this corollary is provided in Figure 1. In the diagram, the initialism CRU means critical-range utilitarianism, and CBU stands for critical-band utilitarianism. An arrow pointing from a theory A to a theory B (that is, $A \to B$) means that theory B is more complete than theory A.

Corollary 1. *Suppose that $\underline{c}, \overline{c} \in \mathbb{R}$ are such that $\underline{c} < \overline{c}$. Then*

$$R_r^{[\underline{c},\overline{c}]} \subsetneq R_r^{(\underline{c},\overline{c}]} \subsetneq R_r^{(\underline{c},\overline{c})} = R_b^{(\underline{c},\overline{c})} = R_b^{(\underline{c},\overline{c}]} = R_b^{[\underline{c},\overline{c})} = R_b^{[\underline{c},\overline{c}]}$$

and

$$R_r^{[\underline{c},\overline{c}]} \subsetneq R_r^{[\underline{c},\overline{c})} \subsetneq R_r^{(\underline{c},\overline{c})} = R_b^{(\underline{c},\overline{c})} = R_b^{(\underline{c},\overline{c}]} = R_b^{[\underline{c},\overline{c})} = R_b^{[\underline{c},\overline{c}]}.$$

This corollary implies that the equivalence between critical-range utilitarianism and critical-band utilitarianism, which is asserted by authors such as Qizilbash (2005, 2007), Rabinowicz (2009, 2012, 2022), Gustafsson (2020), and Williamson (2021), is correct only when the critical band or the critical range Q is an open set.

So far, we have restricted attention to moral quasi-orderings that are based on utilitarianism. However, there is a significant axiological literature that proposes prioritarian principles as an alternative; see Adler (2012) and Adler and Holtug (2019). The fundamental nature of prioritarianism is to assign relatively higher priority to individuals with lower utility levels. Prioritarianism has been examined in the context of population ethics by Brown (2007), Holtug (2022), and Thomas (2022), to name but a few. In view of this interest in prioritarian population theories, we note that it is straightforward to extend our analysis to the case of prioritarianism. All of our formal results remain valid for prioritarianism once we define critical-band prioritarianism and critical-range prioritarianism correspondingly.

5 Critical sets and intervals

An important observation is that both critical-band utilitarianism and critical-range utilitarianism employ a non-degenerate and bounded interval in the definition of the requisite quasi-orderings—and the critical sets associated with these quasi-orderings are non-degenerate and bounded intervals as well. This raises the question whether more general formulations may be possible. In this section, we use a variant of a result established by Blackorby, Bossert, and Donaldson (1996, Lemma 1) and a new observation to illustrate that this structure is a consequence of some mild and quite uncontroversial assumptions.

Our first result in this section differs from that of Blackorby, Bossert, and Donaldson in two respects. First, we employ a slightly weaker property than the strong Pareto principle used by them. Second, the formal statement of our theorem is restricted to the case in which critical sets are distribution-independent. However, as can be verified easily, our observation remains true in the more general setting in which Blackorby, Bossert, and Donaldson's result is valid.

The weakening of strong Pareto that we employ is the following monotonicity condition.

Monotonicity. For all population sizes n and for all utility distributions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, if $u_i \geq v_i$ for all $i \in \{1, \ldots, n\}$ with at least one strict inequality, then u is at least as good as v according to R.

Recall that the property of strong Pareto requires betterness instead of the at-least-as-goodness in its consequent.

Below is our variant of Blackorby, Bossert, and Donaldson's (1996) result, which appears as part of their Lemma 1.

Theorem 4. *Suppose that R is a transitive relation. If R satisfies monotonicity and the critical set C of R contains at least two elements, then C is a non-degenerate interval.*

Theorem 4 provides an intuitively appealing explanation of why critical sets are given by non-degenerate intervals in the population theories considered in this paper. To see why the assumption of boundedness can be plausibly justified as well, consider the following property.

Existence of comparable pairs. For any population size n and for any utility distribution $u = (u_1, \ldots, u_n)$, there exist a sufficiently low utility level a and a sufficiently high utility level b with $a < b$ such that $(u, b)Ru$ and $uR(u, a)$.

An axiom of this kind is suggested by Broome (2009), who writes (Broome, 2009, p. 412)

> "There may be limits: perhaps it is a bad thing to add a person whose life would be miserable, and perhaps a good thing to add a person whose life would be wonderful. But the intuition is that, at least for a range of levels of wellbeing, adding a person within that range has neutral value."

Adding the axiom of existence of comparable pairs to the properties employed in Theorem 4, it follows that the critical set must be bounded in addition to being a non-degenerate interval.

Theorem 5. *Suppose that R is a transitive relation. If R satisfies monotonicity and existence of comparable pairs, and the critical set C for R contains at least two elements, then C is a non-degenerate and bounded interval.*

As mentioned earlier, Broome's intuition states that the critical set does not contain only a single utility level. Hence, according to this theorem, the conjunction of his intuition and his point about limits implies that the critical set must be a non-degenerate and bounded interval.

We note that the results of Theorems 4 and 5 are of significance to axiological theories. This is the case because there are some important theories that do not satisfy strong Pareto but satisfy monotonicity. For example, increases of individual utility above a sufficientarian threshold are morally relevant for some sufficientarian theories. This is what is called the negative thesis (Casal, 2007). An important case in point is that the head-count approach initiated by Frankfurt (1987) satisfies monotonicity but violates strong Pareto because it is possible that the number of those above the threshold might not change even if all individuals are better off. The above theorems imply that the critical set for such a sufficientarian theory must be a non-degenerate and bounded interval, provided that there are at least two critical levels. To the best of our knowledge, there is no sufficientarian theory that violates monotonicity; see Brown (2005), Hirose (2016), and Bossert, Cato, and Kamaga (2022) for sufficientarian theories that satisfy strong Pareto. In sum, a wide range of axiological theories are associated with an interval structure if incompleteness is introduced.

6 Concluding remarks

This paper provides what we think of as some important clarifications. There are some misperceptions in the earlier literature regarding perceived equivalences between various population theories that accommodate non-comparabilities, and our contribution is intended to resolve the ambiguities that may arise from them.

The topic of non-comparability is of growing relevance; there has been a substantial increase in the number of contributions to this topic over the recent past. We hope that our observations serve to further enhance our understanding of the complex issues that cannot but arise in the context of population theories that allow for incomplete moral goodness relations.

This paper shows that there are four distinct types of theories that accommodate non-comparabilities in population ethics. It seems to us that the asymmetry associated with half-open intervals renders the corresponding theories less plausible than those based on open or closed intervals. If this position is adopted, there remain only two possible choices. The first of these is critical-range utilitarianism with an open critical set—which, in view of the openness assumption, is identical to critical-band utilitarianism. The second theory employs closed critical sets, in which case only critical-range utilitarianism is possible. Returning to Broome's intuition of neutrality, the intuition expressed in the following paragraph seems plausible.

> If an individual i is added to a given population and his or her addition has no positive or negative value in itself, then the addition of an individual whose utility level is very close to i's utility level has no positive or negative value in itself.

To see this point, let us consider a utility level c, which is in the critical set. Thus, adding c is neither good nor bad. What Broome's intuition requires is that this cannot be the only critical level, which, in turn, implies that the critical set is an interval that contains c. Now consider a sequence $c^1, c^2, \ldots$ of utility levels that converges to c; that is, c^m approaches c as m approaches infinity. The aforementioned intuition requires that if m is very large, then c^m is a critical level. Arguably, this requirement is plausible—and it implies that the critical set is open. Hence, if this intuition is accepted, a theory with an open critical set is superior to a theory that is based on a closed critical set. Therefore, the above intuition leads us to the conclusion that critical-band utilitarianism (or, equivalently, critical-range utilitarianism) with an open critical set emerges as the most appealing theory among those examined in this paper.

Even if the intuition regarding the role of endpoints is not accepted, the task of further exploring the properties of population theories based on incomplete moral relations is of significant importance. Although theories that assume goodness relations to be complete seem to be well-understood by now, the same cannot be said for moral quasi-orderings, as evidenced by the observation that some demonstrably different theories were perceived to be equivalent in much of the existing literature. A threshold-based theory that accommodates non-comparability is introduced and discussed in Bossert, Cato, and Kamaga (2025).

Appendix A

Proof of Theorem 1. Suppose that R is the critical-band utilitarian quasi-ordering associated with a non-degenerate and bounded interval $Q \subseteq \mathbb{R}$. Consider the population sizes n and m, and the distributions $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$. By definition, if $n = m$,

$$uRv \ \Leftrightarrow \ \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i.$$

Now suppose that $n \neq m$. First, consider the case $n > m$. We show that uRv if and only if (2) and (3) hold. It is straightforward that (2) and (3) together imply uRv. Suppose that uRv. To show that (2) holds, assume that $c = \inf Q$ or $c \in \operatorname{int} Q$. By way of contradiction, suppose that

$$\sum_{i=1}^{n}[u_i - c] \leq \sum_{i=1}^{m}[v_i - c]$$

Since Q is a non-degenerate interval and $c = \inf Q$ or $c \in \operatorname{int} Q$, there exists a sufficiently small positive real number ε such that $(c + \varepsilon) \in Q$. Since $n > m$, it follows that

$$\sum_{i=1}^{n}[u_i - (c+\varepsilon)] = \sum_{i=1}^{n}[u_i - c] - n\varepsilon < \sum_{i=1}^{m}[v_i - c] - m\varepsilon = \sum_{i=1}^{m}[v_i - (c+\varepsilon)].$$

This is a contradiction because uRv and $(c + \varepsilon) \in Q$ together imply

$$\sum_{i=1}^{n}[u_i - (c+\varepsilon)] \geq \sum_{i=1}^{m}[v_i - (c+\varepsilon)].$$

Next, to show that (3) holds, let $c = \sup Q$ and suppose, by way of contradiction, that

$$\sum_{i=1}^{n}[u_i - c] < \sum_{i=1}^{m}[v_i - c].$$

Define the positive real number δ by

$$\delta = \frac{1}{n-m}\left(\sum_{i=1}^{m}[v_i - c] - \sum_{i=1}^{n}[u_i - c]\right).$$

Since Q is a non-degenerate interval and $c = \sup Q$, there exists $\varepsilon \in (0, \delta)$ such that $(c - \varepsilon) \in Q$. We obtain

$$\begin{aligned}
\sum_{i=1}^{m}[v_i - (c-\varepsilon)] - \sum_{i=1}^{n}[u_i - (c-\varepsilon)] &= \sum_{i=1}^{m}[v_i - c] - \sum_{i=1}^{n}[u_i - c] - (n-m)\varepsilon \\
&> \sum_{i=1}^{m}[v_i - c] - \sum_{i=1}^{n}[u_i - c] - (n-m)\delta \\
&= 0
\end{aligned}$$

This is a contradiction because uRv and $(c - \varepsilon) \in Q$ together imply

$$\sum_{i=1}^{n}(u_i - (c-\varepsilon)) \geq \sum_{i=1}^{m}(v_i - (c-\varepsilon)).$$

Finally, we consider the case $n < m$. We show that uRv if and only if (4) and (5) are true. Again, it is straightforward that (4) and (5) together imply uRv. Suppose that uRv. To show that (4) holds, assume that $c = \sup Q$ or $c \in \operatorname{int} Q$. By way of contradiction, suppose that

$$\sum_{i=1}^{n}[u_i - c] \leq \sum_{i=1}^{m}[v_i - c]$$

Since Q is a non-degenerate interval and $c = \sup Q$ or $c \in \operatorname{int} Q$, there exists a sufficiently small positive real number ε such that $(c - \varepsilon) \in Q$. Since $m > n$, it follows that

$$\sum_{i=1}^{n}[u_i - (c - \varepsilon)] = \sum_{i=1}^{n}[u_i - c] + n\varepsilon < \sum_{i=1}^{m}[v_i - c] + m\varepsilon = \sum_{i=1}^{m}[v_i - (c - \varepsilon)].$$

This is a contradiction because uRv and $(c - \varepsilon) \in Q$ together imply

$$\sum_{i=1}^{n}[u_i - (c - \varepsilon)] \geq \sum_{i=1}^{m}[v_i - (c - \varepsilon)].$$

Now, to show that (5) holds, let $c = \inf Q$ and suppose, by way of contradiction, that

$$\sum_{i=1}^{n}[u_i - c] < \sum_{i=1}^{m}[v_i - c].$$

Define the positive real number δ by

$$\delta = \frac{1}{m-n}\left(\sum_{i=1}^{m}[v_i - c] - \sum_{i=1}^{n}[u_i - c]\right).$$

Since Q is a non-degenerate interval and $c = \inf Q$, there exists $\varepsilon \in (0, \delta)$ such that $(c + \varepsilon) \in Q$. Then, we obtain

$$\begin{aligned}\sum_{i=1}^{m}[v_i - (c + \varepsilon)] - \sum_{i=1}^{n}[u_i - (c + \varepsilon)] &= \sum_{i=1}^{m}[v_i - c] - \sum_{i=1}^{n}[u_i - c] - (m-n)\varepsilon \\ &> \sum_{i=1}^{m}[v_i - c] - \sum_{i=1}^{n}[u_i - c] - (m-n)\delta \\ &= 0\end{aligned}$$

This is a contradiction because uRv and $(c + \varepsilon) \in Q$ together imply

$$\sum_{i=1}^{n}(u_i - (c + \varepsilon)) \geq \sum_{i=1}^{m}(v_i - (c + \varepsilon)). \blacksquare$$

Proof of Theorem 2. We prove the case of $Q' = (\underline{c}, \overline{c})$. The proof of the other cases is analogous. Suppose that R and R' are the critical-band utilitarian quasi-orderings associated with $Q = [\underline{c}, \overline{c}]$ and $Q' = (\underline{c}, \overline{c})$, respectively. Consider population sizes n and m arbitrarily and let $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_m) \in \Omega$. By definition, if uRv then $uR'v$ holds. To show that $uR'v$ implies uRv, suppose that $uR'v$. In view of Theorem 1, we assume that $n \neq m$. Define the function $\Delta \colon Q \to \mathbb{R}$ by, for all $c \in Q$,

$$\Delta(c) = \sum_{i=1}^{n} [u_i - c] - \sum_{i=1}^{m} [v_i - c].$$

The function Δ is continuous on Q. From (2) and (4), $uR'v$ implies that

$$\Delta(c) > 0 \text{ for all } c \in \operatorname{int} Q' = Q' = \operatorname{int} Q.$$

Because Δ is continuous on Q, we obtain $\Delta(\overline{c}) \geq 0$ and $\Delta(\underline{c}) \geq 0$. Thus, from Theorem 1, uRv follows. ■

Proof of Theorem 3. First, let R be the critical-range utilitarian relation $R_r^{[\underline{c}, \overline{c}]}$ associated with the closed interval $[\underline{c}, \overline{c}]$. It follows that, for all population sizes n and m and for all distributions $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_m) \in \Omega$, uRv if and only if

$$\text{(i) } n = m \text{ and } \textstyle\sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i$$

or

$$\text{(ii) } n \neq m \text{ and } \textstyle\sum_{i=1}^{n} [u_i - c] > \sum_{i=1}^{m} [v_i - c] \text{ for all } c \in [\underline{c}, \overline{c}].$$

Consider a distribution $u = (u_1, \ldots, u_n)$ and the augmented distribution $(u, \overline{c})$. Then,

$$\sum_{i=1}^{n} [u_i - c] + [\overline{c} - c] - \sum_{i=1}^{n} [u_i - c] = \overline{c} - c \begin{cases} = 0 & \text{if } c = \overline{c} \\ > 0 & \text{if } c \in [\underline{c}, \overline{c}). \end{cases}$$

Thus, by definition, $(u, \overline{c}) N u$, implying that $\overline{c}$ belongs to the critical set. We next show that $\underline{c}$ also belongs to the critical set. For distributions $u = (u_1, \ldots, u_n)$ and $(u, \underline{c}) = (u_1, \ldots, u_n, \underline{c})$, we obtain

$$\sum_{i=1}^{n} [u_i - c] + [\underline{c} - c] - \sum_{i=1}^{n} [u_i - c] = \underline{c} - c \begin{cases} = 0 & \text{if } c = \underline{c} \\ < 0 & \text{if } c \in (\underline{c}, \overline{c}]. \end{cases}$$

Thus, by definition, $(u, \underline{c}) N u$. Finally, for distributions $u = (u_1, \ldots, u_n)$ and $(u, d) = (u_1, \ldots, u_n, d)$ with $d \in (\underline{c}, \overline{c})$, we obtain

$$\sum_{i=1}^{n} [u_i - c] + [d - c] - \sum_{i=1}^{n} [u_i - c] = d - c \begin{cases} > 0 & \text{if } c \in (d, \overline{c}] \\ = 0 & \text{if } c = d \\ < 0 & \text{if } c \in [\underline{c}, d). \end{cases}$$

Thus, by definition, $(u, d) N u$, implying that any $d \in (\underline{c}, \overline{c})$ belongs to the critical set. Consequently, $[\underline{c}, \overline{c}]$ is the critical set for $R_r^{[\underline{c}, \overline{c}]}$.

Now consider $R_r^{(\underline{c},\overline{c})}$, the critical-range utilitarian relation associated with the open interval $(\underline{c},\overline{c})$. We obtain, for all population sizes n and m and for all distributions $u=(u_1,\ldots,u_n), v=(v_1,\ldots,v_m)\in\Omega$, uRv if and only if

$$\text{(i) } n=m \text{ and } \textstyle\sum_{i=1}^n u_i \geq \sum_{i=1}^m v_i$$

or

$$\text{(ii) } n\neq m \text{ and } \textstyle\sum_{i=1}^n [u_i-c] > \sum_{i=1}^m [v_i-c] \text{ for all } c\in Q=(\underline{c},\overline{c}).$$

Consider a distribution $u=(u_1,\ldots,u_n)$ and the augmented distribution $(u,\overline{c})$. Then, for all $c\in(\underline{c},\overline{c})$,

$$\sum_{i=1}^n [u_i-c]+[\overline{c}-c]-\sum_{i=1}^n [u_i-c]=\overline{c}-c>0.$$

Thus, by definition, $(u,\overline{c})Pu$, implying that $\overline{c}$ does not belong to the critical set. Now consider the augmented distribution $(u,\underline{c})$. Then, for all $c\in(\underline{c},\overline{c})$,

$$\sum_{i=1}^n [u_i-c]+[\underline{c}-c]-\sum_{i=1}^n [u_i-c]=\underline{c}-c<0.$$

Thus, by definition, $uP(u,\underline{c})$, implying $\underline{c}$ does not belong to the critical set. Finally, for any augmented distribution (u,d) with $d\in(\underline{c},\overline{c})$, we obtain

$$\sum_{i=1}^n [u_i-c]+[d-c]-\sum_{i=1}^n [u_i-c]=d-c\begin{cases}>0 & \text{if } c\in(d,\overline{c})\\ =0 & \text{if } c=d\\ <0 & \text{if } c\in(\underline{c},d).\end{cases}$$

Thus, by definition, $(u,d)Nu$, implying that d belongs to the critical set. Consequently, $(\underline{c},\overline{c})$ is the critical set for $R_r^{(\underline{c},\overline{c})}$.

The proofs that $(\underline{c},\overline{c}]$ is the critical set for $R_r^{(\underline{c},\overline{c}]}$ and that $[\underline{c},\overline{c})$ is the critical set for $R_r^{[\underline{c},\overline{c})}$ are analogous. ■

Proof of Theorem 4. Although the proof is essentially the same as that used by Blackorby, Bossert, and Donaldson (1996), we provide it for the sake of completeness.

Suppose that R is a transitive relation that satisfies monotonicity and that its critical set C contains at least two elements. To prove that C is an interval, suppose that, by way of contradiction, there exist $a,b\in C$ and $c\in\mathbb{R}$ such that $a<c<b$ and $c\notin C$. Because c is not in the critical set for R, we must have $(u,c)Ru$ or $uR(u,c)$. If $(u,c)Ru$, monotonicity implies $(u,b)R(u,c)$. By transitivity, it follows that $(u,b)Ru$, which contradicts the assumption that b is an element of the critical set. Analogously, if $uR(u,c)$, monotonicity implies $(u,c)R(u,a)$ and, by transitivity, we obtain $uR(u,a)$. Again, this contradicts the assumption that a is an element of the critical set. Thus, C is an interval and, because C contains at least two elements by assumption, it is non-degenerate. ■

Proof of Theorem 5. Suppose that R is a transitive relation that satisfies monotonicity and existence of comparable pairs, and that its critical set C contains at least two elements. By Theorem 4, C is a non-degenerate interval.

To show that C is bounded above, suppose that, by way of contradiction, for all $\overline{K}$, there exists $c \in C$ such that $c \geq \overline{K}$. Let $u \in \Omega$. By existence of comparable pairs, there exists a utility level b such that $(u, b)Ru$. Monotonicity implies that $(u, b')R(u, b)$ for all $b' > b$ and, by transitivity, it follows that $(u, b')Ru$ for all $b' \geq b$. Setting $\overline{K} = b$, it follows that there is no $c \geq \overline{K}$ that is an element of C, a contradiction.

That C is bounded below is proven analogously. ■

Appendix B

We prove that our definition of critical-range utilitarianism is equivalent to that of Rabinowicz (2009).

According to our definition, a relation R_r is a critical-range utilitarian quasi-ordering if there exists a non-degenerate and bounded interval Q such that, for all population sizes n and m and for all utility distributions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$,

$$uR_rv \;\Leftrightarrow\; \left[n = m \;\text{ and }\; \sum_{i=1}^{n} u_i \geq \sum_{i=1}^{m} v_i\right] \;\text{ or }\; \left[n \neq m \;\text{ and }\; \sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c] \;\text{ for all } c \in Q\right].$$

Rabinowicz's (2009) definition states that a relation R_r^* is a critical-range utilitarian quasi-ordering if there exists a non-degenerate and bounded interval Q such that, for all population sizes n and m, and for all utility distributions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$,

$$\text{(i)}\;\; uP_r^*v \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c] \text{ for all } c \in Q \;\text{ and}$$
$$\text{(ii)}\;\; uI_r^*v \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - c] = \sum_{i=1}^{m}[v_i - c] \text{ for all } c \in Q.$$

Suppose that Q is a non-degenerate and bounded interval. To prove that $R_r = R_r^*$, it is sufficient to establish that

$$uP_rv \;\Leftrightarrow\; uP_r^*v \tag{7}$$

and

$$uI_rv \;\Leftrightarrow\; uI_r^*v \tag{8}$$

for all population sizes n and m and for all utility distributions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$.

To establish (7), observe that

$$uP_r v \;\Leftrightarrow\; \left[n = m \;\text{ and }\; \sum_{i=1}^{n} u_i > \sum_{i=1}^{m} v_i\right] \;\text{ or }$$
$$\left[n \neq m \;\text{ and }\; \sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c] \;\text{ for all } c \in Q\right].$$

Note that

$$\left[n = m \;\text{ and }\; \sum_{i=1}^{n} u_i > \sum_{i=1}^{m} v_i\right] \Leftrightarrow \left[n = m \;\text{ and }\; \sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c] \;\text{ for all } c \in Q\right]$$

because c cancels out when $n = m$. From this equivalence, we obtain

$$uP_r v \;\Leftrightarrow\; \left[n = m \;\text{ and }\; \sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c] \;\text{ for all } c \in Q\right] \;\text{ or } \tag{9}$$
$$\left[n \neq m \;\text{ and }\; \sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c] \;\text{ for all } c \in Q\right]. \tag{10}$$

Combining (9) and (10), it follows that

$$uP_r v \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - c] > \sum_{i=1}^{m}[v_i - c] \;\text{ for all } c \in Q$$

and, therefore,

$$uP_r v \;\Leftrightarrow\; uP_r^* v.$$

We complete the proof by showing that (8) is true. By definition,

$$uI_r v \;\Leftrightarrow\; \left[n = m \;\text{ and }\; \sum_{i=1}^{n} u_i = \sum_{i=1}^{m} v_i\right]$$

and

$$uI_r^* v \;\Leftrightarrow\; \sum_{i=1}^{n}[u_i - c] = \sum_{i=1}^{m}[v_i - c] \text{ for all } c \in Q.$$

We first show that if $uI_r^* v$, then the population size n corresponding to $u = (u_1, \ldots, u_n)$ is the same as the population size m corresponding to $v = (v_1, \ldots, v_m)$. By way of contradiction, suppose that $n \neq m$. The statement $uI_r^* v$ is equivalent to

$$\sum_{i=1}^{n} u_i - \sum_{i=1}^{m} v_i = (n - m)c \text{ for all } c \in Q.$$

Since Q is non-degenerate, there exist two values c_1 and c_2 in Q such that $c_1 \neq c_2$. Substituting, we obtain

$$\sum_{i=1}^{n} u_i - \sum_{i=1}^{m} v_i = (n-m)c_1$$

and

$$\sum_{i=1}^{n} u_i - \sum_{i=1}^{m} v_i = (n-m)c_2.$$

Because $n \neq m$, it follows that

$$(n-m)c_1 \neq (n-m)c_2,$$

a contradiction. Therefore, $n = m$. As a result, we have

$$uI_r^*v \Leftrightarrow \left[n = m \text{ and } \sum_{i=1}^{n}[u_i - c] = \sum_{i=1}^{m}[v_i - c] \text{ for all } c \in Q \right]$$

and, because c cancels out when $n = m$, it follows that

$$uI_r^*v \;\Leftrightarrow\; \left[n = m \text{ and } \sum_{i=1}^{n} u_i = \sum_{i=1}^{m} v_i \right]$$

so that

$$uI_rv \;\Leftrightarrow\; uI_r^*v,$$

which completes the proof.

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